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Mean Field Frozen Percolation

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Abstract We define a modification of the Erdős-Rényi random graph process which can be regarded as the mean field frozen percolation process. We describe the behavior of the process using differential equations and investigate their solutions in order to show the self-organized critical and extremum properties of the critical frozen percolation model. We prove two limit theorems about the distribution of the size of the component of a typical frozen vertex.

Keywords Frozen percolation · Random graphs · Smoluchowski coagulation equations

1 Statements

The frozen percolation process on a binary tree was defined by D.J. Aldous in [2]: it is a modification of the percolation process which makes the following informal description mathematically rigorous: we only occupy an edge if both end-vertices are in a finite cluster. The self-organized critical property of this model manifests in the fact that for $t \geq \frac{1}{2}$, which is the critical time of the corresponding percolation process, a typical finite cluster has the distribution of a critical percolation cluster.

I. Benjamini and O. Schramm showed that it is impossible to define a similar modification of the percolation process on $\mathbb{Z}^2$. An explanation of this non-existence result can be found in Sect. 3 of [7].

First we give an informal description of the mean field frozen percolation process: It is a modification of the Erdős-Rényi random graph process: Initially we have a (not necessarily empty) graph on $\lfloor N \cdot m_0(0) \rfloor$ vertices (one should think about N as being large, but the initial mass $m_0(0)$ is fixed), and between every possible pair of vertices, edges appear with rate $\frac{1}{N}$. Simultaneously lightnings strike vertices with rate $\lambda(t)\mu(N)$ at time t and when a vertex is struck, the fire spreads along the edges and burns the connected component of that vertex: that subgraph is removed from the graph, including vertices. Thus the number of vertices

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of the random graph decreases with time. The expressions "burnt", "frozen", "deleted" and "removed" are treated as synonyms in the sequel.

If $\mathcal{V}_k^N(t)$ denotes the number of vertices contained in components of size k in the random graph at time t, then the vector-valued stochastic process $\underline{\mathcal{V}}(t) = (\mathcal{V}_1^N(t), \mathcal{V}_2^N(t), \dots)$ also has the Markov property (the main advantage of the mean field model is that the graph structure of the connected components has no effect on the evolution of component sizes). We are interested in the model when $1 \ll N$.

Denote by $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Definition 1 We fix $m_0(0) \in \mathbb{R}_+$. The mean field frozen percolation process on N vertices is a continuous time Markov process with state space

$$\Omega_N = \left\{ \underline{\mathcal{V}} \in \mathbb{N}_0^{\mathbb{N}} : \sum_{k \geq 1} \mathcal{V}_k \leq \lfloor N \cdot m_0(0) \rfloor,\ \forall k\ \frac{\mathcal{V}_k}{k} \in \mathbb{N}_0 \right\}.$$

We define the coagulation and deletion operators

$$\underline{\mathcal{V}}_{k,l}^{+} := \begin{cases} (\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_k - k, \dots, \mathcal{V}_l - l, \dots, \mathcal{V}_{k+l} + k + l, \dots) & \text{if } k < l, \\ (\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_k - 2k, \dots, \mathcal{V}_{2k} + 2k, \dots) & \text{if } k = l, \end{cases} \tag{1}$$

$$\underline{\mathcal{V}}_k^{-} := (\mathcal{V}_1, \dots, \mathcal{V}_k - k, \dots). \tag{2}$$

Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a positive continuous function and $\mu : \mathbb{N} \to \mathbb{R}_+$. The transition rates of the Markov process are

$$\lambda(\underline{\mathcal{V}} \to \underline{\mathcal{V}}_{k,l}^{+}) = \begin{cases} \frac{1}{N} \cdot \mathcal{V}_k \cdot \mathcal{V}_l & \text{if } k < l, \\ \frac{1}{N} \cdot \frac{\mathcal{V}_k \cdot (\mathcal{V}_k - k)}{2} & \text{if } k = l, \end{cases} \tag{3}$$

$$\lambda(\underline{\mathcal{V}} \to \underline{\mathcal{V}}_k^{-}) = \lambda(t) \cdot \mu(N) \cdot \mathcal{V}_k. \tag{4}$$

Let $v_k^N(t) := \frac{\mathcal{V}_k(t)}{N}$ denote the *mass* of components of size k at time t.

The mean field frozen percolation model is closely related to the mean field forest fire model (discussed in [6]), the only difference in the definition of the Markov process is that in the case of the forest fire model, a burnt component of size k is replaced by k isolated vertices, so that the number of vertices in the random graph remains unchanged. The two models both have the self-organized critical property (and we believe that they are in the same universality class, which means that the theorems of this paper have analogous "forest fire" versions), but the corresponding partial differential equations have an explicit solution in the case of the frozen percolation model which enables us to say more about this model.

$$\mathbf{V} := \left\{ \underline{\mathbf{v}} = \left(v_k\right)_{k=1}^{\infty} :\ v_k \in \mathbb{R},\ v_k \geq 0 \ \text{ and } \ \sum_{k=1}^{\infty} v_k < \infty \right\},$$

$$\mathbf{V}^* := \{\underline{\mathbf{v}} :\ \underline{\mathbf{v}} \in \mathbf{V},\ \exists K < +\infty\ \forall k \geq K\ v_k = 0\}.$$

Definition 2 We consider a sequence of mean field frozen percolation processes with $N \to \infty$, but with the initial state

$$\underline{\mathbf{v}}(0) = \left(v_1^N(0), v_2^N(0), \dots, v_K^N(0), 0, 0, \dots\right) = \left(\frac{\mathcal{V}_1^N(0)}{N}, \frac{\mathcal{V}_2^N(0)}{N}, \dots, \frac{\mathcal{V}_K^N(0)}{N}, 0, 0, \dots\right) \in \mathbf{V}^*$$

and the lightning rate function $\lambda(t)$ fixed (independently of N). Such a sequence is called

- subcritical if $\mu(N) \equiv 1$
- critical if $\frac{1}{N} \ll \mu(N) \ll 1$
- alternating if $\mu(N) = \frac{1}{N}$.

If $v_k(0) = \mathbb{I}_{\{k=1\}} \cdot m_0(0)$ then the initial state is called *monodisperse*, otherwise it is *polydisperse*.

We are going to describe the time evolution of the limit object

$$\lim_{N\to\infty} v_k^N(t) = v_k(t). \tag{5}$$

We introduce differential equations to characterize the limiting component size distributions $v_k(t)$ where $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$. They are modifications of the Smoluchowski coagulation equation with multiplicative rate kernel:

$$\dot{c}_k(t) = \frac{1}{2}\sum_{l=1}^{k-1} l \cdot (k-l) \cdot c_l(t) \cdot c_{k-l}(t) - c_k(t) \sum_{l=1}^{\infty} l \cdot c_l(0) \quad \text{Flory's model,} \tag{6}$$

$$\dot{c}_k(t) = \frac{1}{2}\sum_{l=1}^{k-1} l \cdot (k-l) \cdot c_l(t) \cdot c_{k-l}(t) - c_k(t) \sum_{l=1}^{\infty} l \cdot c_l(t) \quad \text{Stockmayer's model.} \tag{7}$$

If we let $v_k(t) = k \cdot c_k(t)$ then (6) becomes

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t) v_{k-l}(t) - k \cdot v_k(t) \cdot \sum_{k=1}^{\infty} v_k(0). \tag{8}$$

We are going to use the formulation (8) rather than the classical (6).

The differential equations (8) describe the time evolution of $(v_k(t))_{k=1}^{\infty}$ defined by (5) for the dynamical Erdős-Rényi random graph process (see [1]). If we only look at the evolution of the component size vector $\underline{\mathcal{V}}(t)$ in the dynamical Erdős-Rényi random graph model, we get the Marcus-Lushnikov process (see [5]) with multiplicative kernel which is the $\mu(N) \equiv 0$ case of our model (no deletions, only coagulations).

Definition 3 If $(v_k)_{k=1}^{\infty} = \underline{\mathbf{v}} \in \mathbf{V}$ let

$$m_0 := \sum_{k\ge 1} v_k, \qquad m_1 := \sum_{k\ge 1} k v_k, \qquad m_2 := \sum_{k\ge 1} k^2 v_k, \qquad m_3 := \sum_{k\ge 1} k^3 v_k.$$

Remark 1 Our definition of the moments m_n differs from the convention of the literature of the Smoluchowski equation by a shift of indices.

If we define

$$w_k^N(t) := \sum_{l=1}^{k} v_l^N(t) \quad \text{and} \quad \Phi^N(t) := \sum_{l\ge 1} v_l^N(0) - \sum_{l\ge 1} v_l^N(t) = m_0^N(0) - m_0^N(t) \tag{9}$$

then for all k the random function $w_k^N(t)$ is decreasing and $\Phi^N(t)$ (the mass of burnt vertices) is increasing.

It might happen (e.g. in the case of the Erdős-Rényi model) that

$$\theta(t) := \lim_{k\to\infty}\lim_{N\to\infty}\left(m_0^N(t) - w_k^N(t)\right) \neq \lim_{N\to\infty}\lim_{k\to\infty}\left(m_0^N(t) - w_k^N(t)\right) = 0.$$

In this case the mass missing from the small components is contained in a giant component of mass $0 < \theta(t)$.

Definition 4 If $\underline{\mathbf{v}}(t)$ is a solution of (8), we define the gelation time by

$$T^g := \inf\{t : m_1(t) = +\infty\}.$$

It is well-known from the theory of the Smoluchowski coagulation equation that an alternative characterisation of the gelation time is

$$T^g = \inf\{t : m_0(t) < m_0(0)\}.$$

For the solution of (8) the gelation time is $T^g = \frac{1}{m_1(0)}$, the mass of the giant component is $\theta(t) = m_0(0) - m_0(t)$. $\underline{\mathbf{v}}(t)$ undergoes a phase transition:

- For $0 \le t < T^g$ the system is subcritical: $\theta(t) = 0$ and $k \mapsto v_k(t)$ decay exponentially with k.
- For $T^g < t$ the system is supercritical: $\theta(t) > 0$ and $k \mapsto v_k(t)$ decay exponentially with k. Further on: $t \mapsto \theta(t)$ is smooth and strictly increasing with $\lim_{t\to\infty}\theta(t) = m_0(0)$.
- Finally, at $t = T^g$ the system is critical: $\theta(t) = 0$ and

$$\sum_{k=K}^{\infty} v_k(T^g) \asymp K^{-1/2} \quad \text{as } K \to \infty. \tag{10}$$

Our aim is to understand in similar terms the asymptotic behavior of the system when, beside the Erdős-Rényi coagulation mechanism, deletions due to lightnings also take place.

Definition 5 We say that $\underline{\mathbf{v}}(t) = (v_k(t))_{k=1}^{\infty} \in \mathbf{V}$ solves the general frozen percolation equation on $[0, T]$ with initial condition $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$, a continuous nonnegative rate function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ and control function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ if

$$\forall 0 \le s \le t \le T \quad 0 \le \Phi(0) \le \Phi(s) \le \Phi(t) < m_0(0) \tag{11}$$

and for all $k = 1, 2, \dots$ the equations

$$v_k(t) = v_k(0) + \int_0^t \frac{k}{2}\sum_{l=1}^{k-1} v_l(s)v_{k-l}(s) - kv_k(s)\left((m_0(0) - \Phi(s)) + \lambda(s)\right) ds \tag{12}$$

and the inequality

$$\forall t \quad 0 \le \theta(t) := m_0(0) - m_0(t) - \Phi(t) \tag{13}$$

is satisfied.

It is easy to see by induction that the absolutely continuous functions $v_1(t), v_2(t), \dots$ are completely determined by (12), the initial condition $\underline{\mathbf{v}}(0)$ and the functions λ and Φ. The only reason why we do not write

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - kv_k(t)\left((m_0(0) - \Phi(t)) + \lambda(t)\right) \tag{14}$$

instead of (12) is that the increasing function $\Phi(t)$ might have jumps.

There are three versions of the general frozen percolation equation corresponding to the three regimes on Definition 2:

– The *subcritical system of integral equations* are (12) with the extra conditions $\forall t\ 0 < \lambda_{inf} \leq \lambda(t)$ and

$$\Phi(t) \equiv m_0(0) - m_0(t). \tag{15}$$

That is $\theta(t) \equiv 0$ by (13) (no giant components appear due to frequent lightnings) and the equations take on the form

$$v_k(t) = v_k(0) + \int_0^t \frac{k}{2}\sum_{l=1}^{k-1} v_l(s)v_{k-l}(s) - k\cdot v_k(s)m_0(s) - \lambda(s)k\cdot v_k(s)ds. \tag{16}$$

The term $-\lambda(s)k\cdot v_k(s)$ indicates that in the subcritical regime even small components are burnt with a rate proportional to their sizes and $\lambda(s)$.

– The *critical equations* are (12) with the extra conditions $\lambda(t) \equiv 0$ and (15):

$$v_k(t) = v_k(0) + \int_0^t \frac{k}{2}\sum_{l=1}^{k-1} v_l(s)v_{k-l}(s) - k\cdot v_k(s)m_0(s)ds. \tag{17}$$

$\lambda(t) \equiv 0$ indicates that in the critical regime lightnings are not frequent enough to do any harm to small components, but (15) indicates that they are frequent enough to keep the mass of the giant component at zero.

– Let $0 = T_0^b < T_1^b < T_2^b < \cdots$ be a sequence with no accumulation points. Let

$$M(t) := \max\{i : T_i^b < t\} \tag{18}$$

$\underline{\mathbf{v}}(t)$ solves the *alternating equations* with burning times $T_1^b, T_2^b, \ldots$ if

$$\dot{v}_k(t) = \frac{k}{2}\sum_{l=1}^{k-1} v_l(t)v_{k-l}(t) - k\cdot v_k(t)m_0(T_{M(t)}^b). \tag{19}$$

Mind the difference between (8) and (17): in the case of the Erdős-Rényi model the small components are allowed to coagulate with the giant component (which is of size $\theta(t) = m_0(0) - m_0(t)$ by $\Phi(t) \equiv 0$ and (13)), but in the case of the frozen percolation model the giant components are removed at the time of their birth. Using the terminology of the theory of Smoluchowski coagulation equations we might say that in the case of (8) the gel and the sol do react in the post-gel phase (Flory's model, (6)), but in the case of (17) they do not react (Stockmayer's model, (7)). Nevertheless, for $t \leq T^g$ the solutions of (8) and (17) are identical since $m_0(t) = m_0(0)$ in this regime.

The intuitive meaning of (19) is that giant components are removed from the system at the burning times.

Thus (19) is (12) with

$$\theta(t) = m_0(T^b_{M(t)}) - m_0(t), \tag{20}$$

$$\Phi(t) = m_0(0) - m_0(T^b_{M(t)}) = m_0(0) - m_0(t) - \theta(t) = \sum_{j=1}^{M(t)} \theta(T^b_j). \tag{21}$$

Both $\theta(t)$ and $\Phi(t)$ are left-continuous functions of t.

Note that in the case of the (sub)critical frozen percolation equations ((16) and (17)) the fact that $\Phi(t)$ is an increasing function automatically follows by (15):

$$\begin{aligned}\Phi(t) - \Phi(s) &= m_0(s) - m_0(t)\\ &= \sum_{k=1}^{\infty} \int_s^t -\frac{k}{2} \sum_{l=1}^{k-1} v_l(u) v_{k-l}(u) + k \cdot v_k(u) m_0(u) + \lambda(u) \cdot k \cdot v_k(u)\, du\\ &= \lim_{N\to\infty} \int_s^t \sum_{k=1}^{N} \sum_{l=N-k+1}^{\infty} k \cdot v_k(u) v_l(u) + \lambda(u) \cdot k \cdot v_k(u)\, du \geq 0.\end{aligned}$$

Theorem 1

– *For any* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ *and* $0 < \lambda_{inf} \leq \lambda(t)$ (16) *have a unique solution.*
– *For any* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ (17) *have a unique solution.*
– *For any* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ *and any sequence of burning times* (19) *have a unique solution.*

We prove this theorem in Sect. 3.

Definition 6 The solution of the random alternating equations with rate function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathbf{V}$-valued continuous-time Markov process: $\underline{\mathbf{v}}(t)$ evolves deterministically, driven by (19), but the sequence of burning times $T^b_1, T^b_2, \dots$ is random:

$$\lim_{dt\to 0} \frac{1}{dt} \mathbf{P}\big(t \leq T^b_{M(t)+1} \leq t + dt \,\big|\, \mathcal{F}_t\big) = \lambda(t)\theta(t), \tag{22}$$

where $\mathcal{F}_t$ is the natural filtration generated by the process.

In plain words: a lightning strikes and burns the giant component with rate proportional to its size and $\lambda(t)$.

Definition 7

$$\mathcal{W} := \left\{ (w_k)_{k=1}^{\infty} \;:\; 0 \leq w_1 \leq w_2 \leq \dots < +\infty \right\},$$
$$\mathcal{W}^* := \left\{ (w_k)_{k=1}^{\infty} \in \mathcal{W} \;:\; \exists K < +\infty \;\; \forall k \geq K \;\; w_k = w_K \right\}.$$

If $\underline{\mathbf{w}} \in \mathcal{W}$ denote by $m_0 := \sup_k w_k$.

We say that $((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot))$ is a frozen percolation evolution on $[0, T]$ with initial condition $(w_k(0))_{k=1}^{\infty} = \underline{\mathbf{w}} \in \mathcal{W}^*$, or briefly

$$((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot)) \in \mathcal{W}_{\underline{\mathbf{w}}}[0, T]$$

if for all $0 \le t \le T$ we have $(w_k(t))_{k=1}^{\infty} \in \mathcal{W}$, for all k the functions $w_k : [0, T] \to [0, m_0(0)]$ are left-continuous and decreasing, $\Phi : [0, T] \to [0, m_0(0)]$ is left continuous and increasing with initial condition $\Phi(0) = 0$, moreover for all $t \le T$ we have (13).

We define convergence on the space $\mathcal{W}_{\underline{\mathbf{w}}}[0, T]$:

$$((w_k^n(\cdot))_{k=1}^{\infty}, \Phi^n(\cdot)) \to ((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot))$$

as $n \to \infty$ if for all k we have $w_k^n(t) \to w_k(t)$ for all t which is a point of continuity of w_k and $\Phi^n(t) \to \Phi(t)$ for all t which is a point of continuity of Φ.

With this topology the space $\mathcal{W}_{\underline{\mathbf{w}}}[0, T]$ is metrizable, complete and compact.

From the frozen percolation process of Definition 1. one gets a random element of $\mathcal{W}_{\underline{\mathbf{w}}}[0, T]$ by (9). Denote the probability measure on $\mathcal{W}_{\underline{\mathbf{w}}}[0, T]$ corresponding to the process by $\mathbb{P}_N$.

It is easy to check that $((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot)) \in \mathcal{W}_{\underline{\mathbf{w}}}[0, T]$ where $w_k(t) = \sum_{l=1}^{k} v_l(t)$ and $\underline{\mathbf{v}}(t)$ is a solution of the general frozen percolation equation (11) & (12) & (13).

Theorem 2 *We consider a sequence of frozen percolation processes (see Definition 1) with initial state $\underline{\mathbf{v}}^N(0) = \underline{\mathbf{v}}(0) \in \mathbf{V}^*$ and $\lambda(t)$ positive and continuous. Define $w_k^N(t)$ and $\Phi^N(t)$ as in (9). Denote the probability measure on $\mathcal{W}_{\underline{\mathbf{w}}}[0, T]$ corresponding to the process by $\mathbb{P}_N$.*

Then $\mathbb{P}_N$ converges with respect to the weak convergence of probability measures on the polish space $\mathcal{W}_{\underline{\mathbf{w}}}[0, T]$ to a limiting measure $\mathbb{P}$, which depends on the decay rate of $\mu(N)$ in the following way:

- *If $\mu(N) \equiv 1$ then $\mathbb{P}$ is concentrated on the unique solution of (16) with rate function $\lambda(t)$.*
- *If $\frac{1}{N} \ll \mu(N) \ll 1$ then $\mathbb{P}$ is concentrated on the unique solution of (17).*
- *If $\mu(N) = \frac{1}{N}$ then $\mathbb{P}$ is the law of the solution of the random alternating equation (see Definition 6) with rate function $\lambda(t)$.*

We prove the $\mu(N) \equiv 1$ and the $\frac{1}{N} \ll \mu(N) \ll 1$ part of this theorem in Sect. 4. In fact, these proofs are almost identical to the corresponding convergence results of [6], but we present them here as well for the sake of completeness.

We omit the proof of the $\mu(N) = \frac{1}{N}$ part of Theorem 2, but we believe that the methods introduced in Sect. 4. can be easily generalized for this case as well.

If we formally substitute $\lambda(t) \equiv 0$ into (16) or $T^b_{M(t)} \equiv t$ into (19), we get (17). Rigorously:

Theorem 3 *Let $(\underline{\mathbf{v}}^n(t))_{n=1}^{\infty}$ be a sequence of solutions of (16) with the same initial condition $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ where $\lambda_n(t) \to 0$ uniformly as $n \to \infty$. Then for all t and k $\lim_{n\to\infty} v_k^n(t) = v_k(t)$ where $\underline{\mathbf{v}}(t)$ is the solution of (17) with the same initial data. $\lim_{n\to\infty} \Phi_n(t) = \Phi(t)$ uniformly on $[0, \infty)$.*

In plain words: if the rate of lightning is very small in the subcritical equations, then the solution is similar to that of the critical equation. We prove this theorem in Sect. 6.

Theorem 4 *Let $(\underline{\mathbf{v}}^n(t))_{n=1}^{\infty}$ be a sequence of solutions of (19) with the same initial condition $\underline{\mathbf{v}}(0)$ where the sequence of burning times satisfy*

$$\lim_{n\to\infty} \sup_i \{T^b_{i+1}(n) - T^b_i(n)\} = 0.$$

Then for all t and k $\lim_{n\to\infty} v_k^n(t) = v_k(t)$ where $\underline{\mathbf{v}}(t)$ is the solution of (17) with the same initial data. $\lim_{n\to\infty}\Phi_n(t) = \Phi(t)$ *uniformly on* $[0,\infty)$.

In plain words: if the burning times of the alternating equations are very frequent, then the solution is similar to that of the critical equation. We prove this theorem in Sect. 7.

The solution of (17) has the self-organized critical property: for all $T^g \le t$ it has the power-law decay of (10):

Theorem 5 *If $\underline{\mathbf{v}}(t)$ is a solution of (17) with initial condition $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$, then $T^g = \frac{1}{m_1(0)}$, $\Phi(t) = \int_{T^g}^t \varphi_{crit}(s)ds$ where $\varphi_{crit} : [T^g, +\infty) \to R_+$ is positive and continuous, and for all $t \ge T^g$ we have*

$$\lim_{K\to\infty} K^{\frac{1}{2}} \sum_{k=K}^{\infty} v_k(t) = \sqrt{\frac{2\varphi_{crit}(t)}{\pi}}. \tag{23}$$

Definition 8 Let $x^*(t) := \inf\{x : \sum_{k=1}^\infty v_k(t)e^{-kx} < +\infty\}$.

The solutions of our equations have a remarkable rigidity property:

Theorem 6 *If $\underline{\mathbf{v}}(t)$ is the solution of (16) or (19) and $\underline{\tilde{\mathbf{v}}}(t)$ is the solution of (17) with the same initial condition, then for all $t \ge T^g$ and $k \ge 1$ we have*

$$\tilde{v}_k(t) = v_k(t)e^{-kx^*(t)}.$$

The solution of (17) with monodisperse initial condition is well-known (see e.g. [8]) and explicit:

Claim *If $\underline{\mathbf{v}}(t)$ is the solution of (17) with $v_k(0) = \mathbb{I}_{\{k=1\}} \cdot m_0(0)$ then for $t \ge T^g = \frac{1}{m_1(0)} = \frac{1}{m_0(0)}$ we have*

$$v_k(t) = \frac{1}{t}\frac{k^{k-1}}{k!}e^{-k}. \tag{24}$$

That is, for all $T^g \le t$ in the $N \to \infty$ limit, the component size of a uniformly chosen (unburnt) vertex in the critical frozen percolation model has Borel distribution, which is the same as that of a vertex in the Erdős-Rényi graph at $t = T^g$. The Borel distribution $((v_k(1))_{k=1}^\infty$ in (24)) is the distribution of the size of a critical Galton-Watson tree with $POI(1)$ offspring distribution (see [1]).

The same self-similarity phenomenon can be observed in Aldous' frozen percolation model (see [2]) on the binary tree: for $t \ge \frac{1}{2}$, which is the critical time of the percolation process on the binary tree, a typical finite cluster has the distribution of a critical percolation cluster.

The solutions started from a polydisperse initial state are asymptotically self-similar:

Theorem 7 *If $\underline{\mathbf{v}}(t)$ is the solution of the critical equation (17) with $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$, and $v_1(0) > 0$ then*

$$\lim_{t\to\infty} t \cdot v_k(t) = \frac{k^{k-1}}{k!}e^{-k} \quad \textit{and} \quad \lim_{t\to\infty} t \cdot m_0(t) = 1. \tag{25}$$

Theorems 5, 6 and 7 are proved in Sect. 5 using the method of Laplace transforms, which is classical for the Smoluchowski equation with multiplicative kernel. The results (25) and $\frac{v_k(t)}{k} = c_k(t) \asymp k^{-5/2}$ (which is a variant of (23)) are already present in [8], but we believe that our approach based on the notion of the *critical core* of $\underline{\mathbf{v}}(t)$ (defined in Sect. 2) gives new insight into these results about the solution of (17).

In the frozen percolation model on the binary tree, components are frozen (i.e. removed from the system) when their size becomes infinite. The question may arise:

What is the typical size of a frozen component in the mean field process of Definition 1?

In order to precisely formulate this question recall (2) and let

$$\Phi^N([t_1,t_2],k) := \frac{k}{n} \cdot \left|\left\{t \in [t_1,t_2] \,:\, \underline{\mathcal{V}}(t_+) = \underline{\mathcal{V}}_k^-(t_-)\right\}\right|.$$

Thus $\Phi^N([t_1,t_2],k)$ is the mass of burnt components of size k from t_1 to t_2. We have

$$\sum_{k\geq 1} \Phi^N([t_1,t_2],k) = \Phi^N(t_2) - \Phi^N(t_1) =: \Phi([t_1,t_2]).$$

Thus $p_k^N[t_1,t_2] := \frac{\Phi^N([t_1,t_2],k)}{\Phi^N([t_1,t_2])}$, $k = 1, 2, \ldots$ is a random probability distribution for all N and $t_1 < t_2$.

Denote by $|\mathcal{C}^N_{max}(t)|$ the size of the largest component at time t.

Conjecture 1 *If* $\mu(N) = N^{-\alpha}$ *in a critical sequence of frozen percolation processes* (*see Definitions* 1 *and* 2), *where* $0 < \alpha < 1$, *and if we define*

$$\beta(\alpha) := \begin{cases} 2\alpha & \text{if } \alpha \leq \frac{1}{3}, \\ \frac{\alpha+1}{2} & \text{if } \alpha \geq \frac{1}{3} \end{cases} \tag{26}$$

then for every $T^g < t$ *we have*

$$\lim_{N\to\infty} \frac{\log(\mathbf{E}(m_1^N(t)))}{\log(N)} = \alpha, \tag{27}$$

$$\lim_{N\to\infty} \frac{\log(\mathbf{E}(m_2^N(t))) - \log(\mathbf{E}(m_1^N(t)))}{\log(N)} = \beta(\alpha), \tag{28}$$

$$\lim_{N\to\infty} \frac{\log(\mathbf{E}(|\mathcal{C}^N_{max}(t)|))}{\log(N)} = \beta(\alpha). \tag{29}$$

Moreover for every $\underline{\mathbf{v}}(0)$, $T^g < t_1 < t_2$ *and* α *there exists a non-defective probability distribution function* $F : (0,\infty) \to (0,1)$, $\lim_{x\to 0_+} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$ *such that for all* $x \in \mathbb{R}_+$ *we have*

$$\lim_{N\to\infty} \sum_{k\geq 1} \mathbb{I}[k \leq x N^{\beta(\alpha)}] \cdot p_k^N[t_1,t_2] = F(x). \tag{30}$$

In plain words we might say that after gelation the typical component size of a frozen vertex and the size of the largest component is of order $N^{\beta(\alpha)}$. This conjecture is supported by heuristic arguments, computer simulations and Theorems 8 and 9 below. For $0 < \alpha < \frac{1}{3}$ the model is conjectured to behave similarly to the subcritical case described in Theorem 8, whereas for $\frac{1}{3} < \alpha < 1$ it is conjectured to behave similarly to the alternating case described in Theorem 9. Note that $\beta(\frac{1}{3}) = \frac{2}{3}$ and $N^{\frac{2}{3}}$ is the order of the size of the largest component in the critical Erdős-Rényi random graph.

Theorem 8 *If $\underline{\mathbf{v}}^\lambda(t)$ is the solution of (16) with rate function $\lambda(t) \equiv \lambda$ and $\underline{\mathbf{v}}^\lambda(0) = \underline{\mathbf{v}}(0) \in \mathbf{V}^*$ then there is a constant C that depends only on the initial data and T such that for all $0 < \lambda \le 1$ and $\frac{1}{m_1(0)} < t \le T$ we have*

$$|\varphi_\lambda(t) - \varphi_{crit}(t)| \le C\lambda, \tag{31}$$

where

$$\frac{d}{dt}\Phi_\lambda(t) = \varphi_\lambda(t) = \lambda m_1^\lambda(t). \tag{32}$$

Moreover if we define the random variable $Y_\lambda(t)$ to have distribution

$$\mathbf{P}\left(Y_\lambda(t) = k\right) = \frac{\lambda \cdot k \cdot v_k^\lambda(t)}{\varphi_\lambda(t)} = \frac{k \cdot v_k^\lambda(t)}{m_1^\lambda(t)}$$

then

$$\lim_{\lambda \to 0} \mathbf{P}\left(\frac{\lambda^2}{2\varphi_{crit}(t)} Y_\lambda(t) < x\right) = \int_0^x \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{y}} e^{-y} dy. \tag{33}$$

In plain words: for any $t > T^g$ the distribution of the size-biased sample from the component-size distribution $\underline{\mathbf{v}}^\lambda(t)$ rescaled by λ^{-2} converges in distribution to a $\Gamma(\frac{1}{2}, 1)$ distribution as $\lambda \to 0$. We prove this theorem in Sect. 7.

The relevance of Theorem 8 to Conjecture 1 is the following: if we consider a sequence of subcritical frozen percolation models (see Definition 2) with $\lambda(t) \equiv \lambda$ then by Theorem 2 we get

$$\begin{aligned} \lim_{dt \to 0} \lim_{N \to \infty} p_k^N[t, t+dt] &= \lim_{dt \to 0} \frac{\Phi_\lambda([t, t+dt], k)}{\Phi_\lambda([t, t+dt])} \\ &= \lim_{dt \to 0} \frac{\int_t^{t+dt} \lambda \cdot k \cdot v_k^\lambda(s)\, ds}{\int_t^{t+dt} \sum_{l=1}^\infty \lambda \cdot l \cdot v_l^\lambda(s)} = \frac{k \cdot v_k^\lambda(t)}{m_1^\lambda(t)} = \mathbf{P}\left(Y_\lambda(t) = k\right). \end{aligned}$$

If we let $\lambda \to 0$ then by (31) and (32) we get $m_1^\lambda(t) \asymp \lambda^{-1}$ which is a "subcritical" version of (27), $\frac{m_2^\lambda(t)}{m_1^\lambda(t)} = \mathbf{E}(Y_\lambda(t)) \asymp \lambda^{-2}$ corresponds to $\beta(\alpha) = 2\alpha$ in (28), and (33) is a version of (30).

Theorem 9 *Let $\underline{\mathbf{v}}^\lambda(t)$ denote the solution of the random alternating equations (see Definition 6) with a constant rate function $\lambda(t) \equiv \lambda$.*

Let $\delta(\lambda)$ be a function satisfying $\lambda^{-\frac{1}{2}} \ll \delta(\lambda) \ll 1$ as $\lambda \to \infty$.

Recalling (18) and (20) let

$$\Phi_\lambda(t, x) := \sum_{j=1}^{M(t)} \theta^\lambda(T_j^b) \mathbb{I}[\theta^\lambda(T_j^b) > x]$$

be the random mass of frozen giants of size at least x. Then

$$\lim_{\lambda\to\infty} \frac{\Phi_\lambda(t+\delta(\lambda), 2\sqrt{\frac{\varphi_{crit}(t)}{\lambda}}x) - \Phi_\lambda(t, 2\sqrt{\frac{\varphi_{crit}(t)}{\lambda}}x)}{\delta(\lambda)\varphi_{crit}(t)} = \int_x^\infty \frac{4}{\sqrt{\pi}} y^2 e^{-y^2} dy \tag{34}$$

in probability.

We prove this theorem in Sect. 7.

The heuristic meaning of this theorem is the following: if we pick a vertex uniformly from all vertices that were frozen between t and $t+\delta(\lambda)$ and denote the mass of the giant component of that vertex by $Z_\lambda(t)$, then the distribution of $\frac{1}{2}\sqrt{\frac{\lambda}{\varphi_{crit}(t)}}Z_\lambda(t)$ converges to a size-biased Rayleigh distribution (see Definition 15) as $\lambda\to\infty$. Thus the typical mass of a frozen giant is of order $\lambda^{-\frac{1}{2}}$, which suggests that if $\mu(N) = \frac{N^\varepsilon}{N}$ (that is $\alpha = 1-\varepsilon$ in Conjecture 1) then the typical size of a frozen component is of order $(N^\varepsilon)^{-\frac{1}{2}} \cdot N = N^{1-\frac{1}{2}\varepsilon}$, that is $\beta(\alpha) = \frac{\alpha+1}{2}$. Equation (34) is the "alternating" version of (30).

The critical frozen percolation model has an extremum property compared to the sub-critical and alternating models (see Definition 2): if each burnt/frozen vertex produces profit at a rate $\frac{1}{N}$\$ per time unit after it has been frozen, but each lightning (even the ones hitting burnt vertices) costs $\frac{1}{N\cdot m_0(0)}$\$, then asymptotically (as $N\to\infty$) the critical model is the best choice if we want to maximize our profit on $[0,T]$. We reformulate this extremum principle in terms of the differential equations (16), (17), (19).

The asymptotic value of our profit produced by burnt vertices as $N\to\infty$ is $\int_0^T \Phi(t)dt$ according to Theorem 2. The asymptotic cost of lightnings is $\int_0^T \lambda(t)dt$ for the solution of (16), but it is zero for (17) and (19), since the price we have to pay for the lightnings vanishes in the case of critical and alternating models as $N\to\infty$.

Theorem 10 *We fix $\underline{\mathbf{v}}(0)\in\mathbf{V}^*$. Let $\underline{\mathbf{v}}^{crit}(t)$ denote the solution of (17) with initial condition $\underline{\mathbf{v}}(0)$ and let $\underline{\mathbf{v}}^{sub}$ denote the solution of (16) with lightning rate function $\lambda(t)$ and the same initial condition. Then for any $T>0$*

$$\int_0^T \Phi^{sub}(t)dt - \int_0^T \lambda(t)\,dt \le \int_0^T \Phi^{crit}(t)dt - \int_0^T 0\,dt. \tag{35}$$

If $\underline{\mathbf{v}}^{alt}(t)$ denotes the solution of (19) with an arbitrary sequence of burning times and initial condition $\underline{\mathbf{v}}(0)$ then

$$\int_0^T \Phi^{alt}(t)dt \le \int_0^T \Phi^{crit}(t)dt. \tag{36}$$

Remark 2 Let $T > T^g = \frac{1}{m_1(0)}$ and $\varepsilon > 0$. For a suitable choice of $\lambda(t)$ we have

$$\int_0^T \Phi^{sub}(t)dt - (1-\varepsilon)\int_0^T \lambda(t)\,dt > \int_0^T \Phi^{crit}(t)dt - (1-\varepsilon)\int_0^T 0\,dt. \tag{37}$$

For a suitable choice of burning times

$$\int_0^T \Phi^{alt}(t)dt + \varepsilon\Phi^{alt}(T) > \int_0^T \Phi^{crit}(t)dt + \varepsilon\Phi^{crit}(T). \tag{38}$$

The idea that the critical *forest fire* model solves a variational problem is already present in [3].

2 Definitions, Transformations

We consider a solution of the general frozen percolation equation (see Definition 5).

Denote the Laplace transform (generating function) of $\underline{\mathbf{v}}(t)$ by

$$V(t,x) := \sum_{k=1}^{\infty} v_k(t) e^{-kx} \tag{39}$$

for $x > 0$. Then $V(t,0) = V(t,0_+) = m_0(t)$ and by dominated convergence for $x > 0$ (16) is transformed into

$$V(t,x) = V(0,x) + \int_0^t V'(s,x)(-V(s,x) + (m_0(0) - \Phi(s)) + \lambda(s))ds. \tag{40}$$

In the sequel we denote the derivative of functions $f(t,x)$ with respect to the time and space variables by $\dot{f}(t,x)$ and $f'(t,x)$, respectively.

Let

$$U(t,x) := V(t,x) - (m_0(0) - \Phi(t)). \tag{41}$$

Thus (40) is transformed into

$$U(t,x) = U(0,x) + \int_0^t -U(s,x)U'(s,x) + \lambda(s)U'(s,x)ds + \Phi(t). \tag{42}$$

Since $V(t,\cdot)$ is a Laplace transform we have

$$U(t,0) = -\theta(t), \qquad U'(t,0) = -m_1(t), \qquad \lim_{x\to\infty} U(x) = -m_0(0) + \Phi(t) \tag{43}$$

and U is a monotone decreasing convex function of the variable x for every t.

Definition 9 Denote by $X(t,u)$ the inverse function of $U(t,x)$ with respect to x, that is $U(t, X(t,u)) = u$.

The domain of $X(t,u)$ in the variable u is $(-m_0(t) + \Phi(t), -\theta(t)]$.

$$X(t, -\theta(t)) = 0. \tag{44}$$

The notion of $X(t,\cdot)$ and a version of the following lemma is already present in [8].

Lemma 1 *If $X(t,u)$ is defined using a solution of the general frozen percolation equation then the following identity holds*:

$$X(t,u) = X(0, u - \Phi(t)) + t \cdot (u - \Phi(t)) - \int_0^t \lambda(s)ds + \int_0^t \Phi(s)ds. \tag{45}$$

Proof We fix an $x_{min} > 0$. For any $x \geq x_{min}$ we have

$$|U(t,x)| \leq m_0(0), \qquad |U'(t,x)| \leq \frac{m_0(t)}{x_{min}}, \qquad |U''(t,x)| \leq \frac{m_0(t)}{x_{min}^2}, \tag{46}$$

moreover $\sup_{0\leq t\leq T} \lambda(t) < +\infty$. For an $x(0) > x_{min}$ denote by $x(t)$ the solution of the integral equation

$$x(t) = x(0) + \int_0^t U(s,x(s)) - \lambda(s)ds. \tag{47}$$

This equation is well-posed on the domain $x(t) \geq x_{min}$, since $U(s,x) - \lambda(s)$ is bounded and Lipschitz-continuous in x.

Moreover

$$x(t+dt) - x(t) = \mathcal{O}(dt), \qquad |U(t,x(t)) - U(t,x(t+dt))| = \mathcal{O}\left(\frac{dt}{x_{min}}\right).$$

If we differentiate (42) w.r.t. x we get $|U'(t+dt,x) - U'(t,x)| = \mathcal{O}(\frac{dt}{x_{min}^2})$.

$$\begin{aligned}
&U(t+dt,x(t+dt)) - U(t,x(t)) - (\Phi(t+dt) - \Phi(t)) \\
&\quad= \big(U(t+dt,x(t+dt)) - U(t,x(t+dt))\big) \\
&\qquad+ \big(U(t,x(t+dt)) - U(t,x(t))\big) - (\Phi(t+dt) - \Phi(t)) \\
&\quad= \int_t^{t+dt} -U(s,x(t+dt))U'(s,x(t+dt)) + \lambda(s)U'(s,x(t+dt))ds \\
&\qquad+ U'(t,x(t+dt)) \int_t^{t+dt} U(s,x(s)) - \lambda(s)ds + \mathcal{O}\left(\frac{dt^2}{x_{min}^2}\right) \\
&\quad= \int_t^{t+dt} U(s,x(t+dt))\big(U'(t,x(t+dt)) - U'(s,x(t+dt))\big)ds \\
&\qquad+ \int_t^{t+dt} U'(t,x(t+dt))\big(U(s,x(s)) - U(s,x(t+dt))\big)ds \\
&\qquad+ \int_t^{t+dt} \lambda(s)\big(U'(s,x(t+dt)) - U'(t,x(t+dt))\big)ds + \mathcal{O}\left(\frac{dt^2}{x_{min}^2}\right) = \mathcal{O}\left(\frac{dt^2}{x_{min}^2}\right).
\end{aligned}$$

Thus $U(t,x(t)) = U(0,x(0)) + \Phi(t)$, and if we substitute this back into (47), we get

$$x(t) = x(0) + tU(0,x(0)) + \int_0^t \Phi(s)ds - \int_0^t \lambda(s)ds.$$

By the definition of $X(t,u)$ we have $X(t,U(t,x(t))) = x(t)$, and by substituting

$$u = U(0,x(0)) + \Phi(t)$$

we obtain (45). □

Since $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$, $V(0,x)$ is well-defined and analytic for all $x \in \mathbb{R}$, thus $X(0,u)$ can be analytically extended to $(-m_0(0),+\infty)$. Equation (45) makes it possible to extend $X(t,u)$

to $(-m_0(0)+\Phi(t),+\infty)$ analytically. The extended $X(t,u)$ is a strictly convex function of the u variable. If we differentiate (45) w.r.t. u, we get

$$X'(t,u)=X'(0,u-\Phi(t))+t. \tag{48}$$

Definition 10 Define $F(t,w)$ by the identity

$$F(t,-X'(t,u))=-u. \tag{49}$$

Thus $-F(t,w)$ is the inverse function of $-X'(t,u)$. If $\hat{X}$ denotes the Legendre-transform of X w.r.t. the variable u, then

$$G(t,w):=\hat{X}(t,-w)=-\min_u\{wu+X(t,u)\}=wF(t,w)-X(t,-F(t,w)). \tag{50}$$

Let

$$E(t,w)=G''(t,w)=F'(t,w). \tag{51}$$

We call $E(t,\cdot)$ the *critical core* of $\underline{\mathbf{v}}(t)$. If we use the extended definition of X then $G(t,w)$ is well-defined and analytic for all $w>-t$.

We have

$$F\left(t,-\frac{1}{U'(t,x)}\right)=-U(t,x)\quad\text{and}\quad E\left(t,-\frac{1}{U'(t,x)}\right)=\frac{(-U'(t,x))^3}{U''(t,x)}. \tag{52}$$

It follows from the properties of the Legendre-transformation and (45) that

$$G(t,w)=G(0,w+t)-w\cdot\Phi(t)-\int_0^t\Phi(s)ds+\int_0^t\lambda(s)ds, \tag{53}$$

$$F(t,w)=F(0,w+t)-\Phi(t), \tag{54}$$

$$E(t,w)=E(0,w+t). \tag{55}$$

$G(t,\cdot)$ is strictly convex and G determines X uniquely since the Legendre-transformation is invertible. Define

$$w^*(t):=-X'(t,0)\iff F(t,w^*(t))=0\iff \operatorname*{argmin}_w G(t,w)=w^*(t), \tag{56}$$

$$X(t,0)=0\implies G(t,w^*(t))=0\implies \forall w\; G(t,w)\geq 0, \tag{57}$$

$$\theta(t)=0\quad\implies\quad w^*(t)=\frac{1}{m_1(t)}\geq 0, \tag{58}$$

$$x^*(t)=\inf\left\{x:\sum_{k=1}^{\infty}v_k(t)e^{-kx}<+\infty\right\}=\min_u X(t,u)=X(t,-F(t,0))=-G(t,0). \tag{59}$$

3 The frozen percolation equations are well-posed

Lemma 2 *The alternating equation* (19) *is well-posed.*

Proof If we are given the sequence of burning times $0 < T_1^b < T_2^b < \cdots$ the solution of (19) can be uniquely constructed by using induction on i: if we already have the solution on $[0, T_i^b]$, then we are given $m_0(T_i^b)$, so we can uniquely solve the sequence of ordinary differential equations (19) for $v_1, v_2, \ldots$ on $[T_i^b, T_{i+1}^b]$ by repeatedly applying the Picard-Lindelöf theorem, since the equation for v_k only contains $v_1, \ldots, v_k$ on its right-hand side. □

Lemma 3 *The solution of the integral equations* (16) *is unique for every initial condition* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ *if* $\lambda(t)$ *is nonnegative and continuous.*

Remark 3 Choosing $\lambda(t) \equiv 0$ implies the uniqueness of the solutions of (17).

Proof In order to prove the uniqueness of the solution of (16), we only have to prove that given two solutions with the same initial condition, the function $\Phi(t) = m_0(0) - m_0(t)$ determined by the two solutions is the same, because $m_0(t)$ and (16) determines $v_k(t)$ for all k uniquely. For a solution $\underline{\mathbf{v}}(t)$ of (16) we can define U by (41), then X by Definition 9, which satisfies (45) and the G of Definition 10 satisfies (53).

Assume that G_1 and G_2 are obtained this way from two solutions of (16) with the same initial condition $G(0, w)$. Let $\tilde{G} = G_1 - G_2$ and $\tilde{\Phi} = \Phi_1 - \Phi_2$. Then

$$\tilde{G}(t, w) = -w \cdot \tilde{\Phi}(t) - \int_0^t \tilde{\Phi}(s)ds.$$

Now by (15) we have $\theta(t) = 0$, thus (44) $\Longrightarrow$ $X(t, 0) = 0$, and (57) $\Longrightarrow$ $\min_w G_1(t, w) = \min_w G_2(t, w) = 0$ and (58) $\Longrightarrow$ $w_i^*(t) := \mathrm{argmin}_w G_i(t, w) \geq 0$ for $i = 1, 2$, thus we have $\tilde{G}(t, w_1^*(t)) \leq 0$ and $\tilde{G}(t, w_2^*(t)) \geq 0$. Thus $\tilde{\Phi}(t)$ and $\int_0^t \tilde{\Phi}(s)ds$ cannot have the same sign. But if (t_1, t_2) is a maximal interval such that for $t_1 < t < t_2$ we have $\int_0^t \tilde{\Phi}(s)ds > 0$ then $\int_0^{t_1} \tilde{\Phi}(s)ds = 0$ and

$$t \in [t_1, t_2] \Longrightarrow \int_0^t \tilde{\Phi}(s)ds \geq 0 \Longrightarrow \tilde{\Phi}(t) \leq 0 \Longrightarrow \int_{t_1}^t \tilde{\Phi}(s)ds \leq 0$$

which contradicts the definition of t_1 and t_2. Thus $\int_0^t \tilde{\Phi}(s)ds \leq 0$ for all t and interchanging the roles of G_1 and G_2 we get $\int_0^t \tilde{\Phi}(s)ds \equiv 0$, so $\Phi_1(t) \equiv \Phi_2(t)$. □

Lemma 4 *If we find a function $\varphi(t)$ such that defining $\Phi(t) := \int_0^t \varphi(s)ds$ and $G(t, w)$ by* (53) *we have*

$$\min_w G(t, w) = 0 \quad \text{and} \quad w^*(t) = \operatorname*{argmin}_w G(t, w) \geq 0 \tag{60}$$

for all t, then the solution of (12) *with the same $\lambda(\cdot)$, $\Phi(\cdot)$ and initial condition satisfies* (16).

Proof Since the Legendre-transformation is invertible, from (60) we get

$$X(t, 0) = 0 \quad \text{and} \quad X'(t, 0) \leq 0.$$

$X(t, u)$ is strictly decreasing for $u < 0$, thus it is the inverse function of an $U(t, x)$ satisfying $U(t, 0) = 0$. If we plug $\Phi(\cdot)$ into (12) then we get $\theta(t) = -U(t, 0) = 0$, therefore (15) is satisfied. □

Lemma 5 *The* Φ *of the unique solution of* (17) *is*

$$\Phi(T)=\begin{cases}0 & \text{if } t\le T^g,\\ F(0,T) & \text{if } t\ge T^g,\end{cases} \tag{61}$$

where $T^g=\frac{1}{m_1(0)}$.

$$\int_0^T \Phi(t)dt=\begin{cases}0 & \text{if } T\le T^g,\\ G(0,T) & \text{if } T\ge T^g.\end{cases} \tag{62}$$

Proof The solution is unique according to Lemma 3 and to prove its existence we only have to find a function $\varphi(t)$ that satisfies the criteria of Lemma 4 (with $\lambda(t)\equiv 0$). We will show that

$$\varphi(t)=\mathbb{I}\left[t\ge\frac{1}{m_1(0)}\right]E(0,t) \tag{63}$$

does the job. For $t\le T^g$ this is trivial by looking at (53): $G(t,w^*(t))=0$ and $w^*(t)=\frac{1}{m_1(0)}-t\ge 0$ if $\Phi(t)\equiv 0$.

We will show that for $t\ge T^g$ we have $G(t,0)\equiv 0$ and $F(t,0)\equiv 0$, that is $w^*(t)\equiv 0$. $F(0,T^g)=G(0,T^g)=0$ by (57) and $w^*(0)=\frac{1}{m_1(0)}=T^g$. $F(t,0)=0$ follows from (54) and

$$\Phi(t)=\int_0^t\varphi(s)ds=\int_{T^g}^t E(0,s)ds=F(0,t)-F(0,T^g)=F(0,t).$$

By (53) we have

$$G(t,0)=G(0,t)-\int_0^t\Phi(s)ds=\int_{T^g}^t F(0,s)ds-\int_{T^g}^t F(0,s)ds=0. \qquad\square$$

The well-posedness of the integral equation (17) implies that of the corresponding differential equation, since $m_0(0)-\Phi(t)=m_0(t)$ is a continuous function of t, thus $v_k(t)$ are differentiable.

We have shown that the solution of (17) has infinite first moment after the gelation time: $\frac{1}{w^*(t)}=m_1(t)=+\infty$ for all $t\ge T^g$.

Definition 11 Let $E(0,w)$ denote the critical core of $\underline{\mathbf{v}}(0)$ (see Definition 10).

For $\frac{1}{m_1(0)}\le w_1\le w_2$ define

$$E_{inf}(w_1,w_2):=\min_{w_1\le w\le w_2}E(0,w)\quad\text{and}\quad E_{sup}(w_1,w_2):=\max_{w_1\le w\le w_2}E(0,w),$$

$$E_{sup}:=E_{sup}\left(\frac{1}{m_1(0)},+\infty\right),\qquad E_{inf}(w):=E_{inf}\left(\frac{1}{m_1(0)},w\right).$$

Lemma 6 *If* $w\ge\frac{1}{m_1(0)}$ *then the inequalities*

$$\frac{m_1(0)}{m_2(0)}\frac{1}{w^2}\le E(0,w)\le\frac{1}{w^2} \tag{64}$$

hold. Thus $E_{sup}\le m_1(0)^2$ *and* $E_{inf}(w)\ge\frac{m_1(0)}{m_2(0)}\frac{1}{w^2}$.

For all $w \geq \frac{1}{m_1(0)}$ *we have*

$$\left|E'(0, w)\right| \leq 4m_2(0)^2 m_3(0) =: D \tag{65}$$

which implies

$$E_{sup}(w_1, w_2) - E_{inf}(w_1, w_2) \leq D \cdot (w_2 - w_1). \tag{66}$$

Remark 4 If $m_1(0) = m_2(0)$ then the upper and lower bounds in (64) coincide. This can only happen if $v_k(0) = m_1(0) \cdot \mathbb{I}[k = 1]$, this is the case known as the monodisperse initial condition (the initial graph has no edges).

Proof Let $U(x) := U(0, x)$. Recalling (52) $E(0, -\frac{1}{U'(x)}) = \frac{(-U'(x))^3}{U''(x)}$ holds. The upper bound of (64) follows from $-U'(x) \leq U''(x)$, and $-U'(x)\frac{m_2(0)}{m_1(0)} \geq U''(x)$ holds because $\log(-U'(x))$ is a convex function, thus $\frac{U''(x)}{U'(x)} \geq \frac{U''(0)}{U'(0)} = \frac{m_2(0)}{-m_1(0)}$. The bound on the Lipschitz constant (65) follows from

$$\left|E'\left(0, -\frac{1}{U'}\right)\right| = \left|\frac{(U')^5 U'''}{(U'')^3} - 3\frac{(U')^4}{U''}\right| \leq \left|(U')^2 U'''\right| + 3\left|(U')^3\right| \leq 4m_2(0)^2 m_3(0). \qquad \square$$

Now we turn our attention to the subcritical equation (16). We assume $\lambda(t) > 0$ for all t. If we substitute $x = 0$ into the differential equation (42) and assume $|U'(t, 0)| < +\infty$ then (formally) we get

$$\dot{\Phi}(t) = \varphi(t) = -U'(t, 0) \cdot \lambda(t) = m_1(t)\lambda(t) = \frac{\lambda(t)}{w^*(t)}.$$

Definition 12 If $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ and $\lambda(t)$ is a positive continuous function then the subcritical control differential equation for $w^*(t)$ is

$$\dot{w}^*(t) = \frac{\lambda(t)}{w^*(t)E(0, t + w^*(t))} - 1 \tag{67}$$

with initial condition $w^*(0) = \frac{1}{m_1(0)} = T^g$.

Lemma 7 *The subcritical control differential equation is well-posed and the function*

$$\varphi(t) := \frac{\lambda(t)}{w^*(t)}$$

(*where* $w^*(t)$ *is the solution of* (67) *with* $w^*(0) = \frac{1}{m_1(0)}$) *satisfies the criteria of Lemma* 4, *which implies the existence of solutions to* (16).

Proof We prove the statement of the lemma on $[0, T]$. The Picard-Lindelöf theorem and the Lipschitz-continuity property (65) guarantee the existence and uniqueness of the solution of (67) before the graph of the solution exits

$$\{(t, w^*) : 0 \leq t \leq T,\ w^*_{min} \leq w^* \leq w^*_{max},\ w^* + t \geq w^*(0)\} \tag{68}$$

for some $0 < w^*_{min} < w^*_{max} < +\infty$.

Let $\lambda_{inf} := \inf_{0\le t\le T} \lambda(t)$, $\lambda_{sup} := \sup_{0\le t\le T} \lambda(t)$. From (67) and a "forbidden region"-type argument we get that $w^*(t)+t \ge w^*(0)$ and $w^*(t) \ge \min\{\frac{\lambda_{inf}}{E_{sup}}, w^*(0)\}$, since

$$w^*(t) > 0 \implies \frac{d}{dt}(w^*(t)+t) \ge 0,$$
$$w^*(t)+t \ge w^*(0) \implies E(0, t+w^*(t)) \le E_{sup},$$

thus $w^*(t) < \frac{\lambda_{inf}}{E_{sup}} \implies \dot{w}^*(t) > 0$.

Now we prove that $w^*(t)$ cannot grow too fast using the lower bound of (64). $w^*(t) \le y(t)$ where $y(0) = w^*(0) = T^g$ and

$$\dot{y}(t) = \lambda_{sup} \frac{m_2(0)}{m_1(0)} \frac{(y(t)+t)^2}{y(t)} \le \lambda_{sup} \frac{m_2(0)}{m_1(0)} \left(\frac{T^g+t}{T^g}\right) \cdot (y(t)+t)$$

since $y(t)$ is increasing. Thus $\dot{y}(t) \le a \cdot y(t) + b$ for some a and b depending only on the initial data, the function $\lambda(t)$ and T. Thus

$$w^*(t) \le w^*(0)e^{at} + \frac{b}{a} \cdot (e^{at} - 1).$$

Now we can see that the graph of the solution of (67) indeed doesn't exit (68) until $t = T$ if we define

$$w^*_{min} = \min\left\{\frac{\lambda_{inf}}{E_{sup}}, T^g\right\} \quad \text{and} \quad w^*_{max} := \left(T^g + \frac{b}{a}\right)e^{aT}. \tag{69}$$

Now we prove that $\varphi(t) := \frac{\lambda(t)}{w^*(t)}$ satisfies the criteria of Lemma 4 by showing that

$$G(t, w^*(t)) \equiv 0 \quad \text{and} \quad F(t, w^*(t)) \equiv 0.$$

This holds for $t = 0$, so it suffices to check $\frac{d}{dt}G(t, w^*(t)) \equiv 0$ and $\frac{d}{dt}F(t, w^*(t)) \equiv 0$. Using (54)

$$\frac{d}{dt}F(t, w^*(t)) = E(0, t + w^*(t)) \cdot \left(1 + \frac{\lambda(t)}{w^*(t)E(0, t+w^*(t))} - 1\right) - \frac{\lambda(t)}{w^*(t)} = 0.$$

If we combine $F(t, w^*(t)) \equiv 0$ with (54) we get

$$F(0, t + w^*(t)) = \Phi(t). \tag{70}$$

It is straightforward to verify $\frac{d}{dt}G(t, w^*(t)) \equiv 0$ by using (53) and (70). □

This completes the proof of the well-posedness of (16).

4 Proof of Theorem 2

We consider the sequence $\mathbb{P}_N$ of probability measures on the compact space $\mathcal{W}_{\underline{\mathbf{w}}}[0, T]$. From Prokhorov's theorem it follows that any subsequence of the measures $\mathbb{P}_N$ contains a sub-subsequence that converges weakly to a limiting measure on $\mathcal{W}_{\underline{\mathbf{w}}}[0, T]$.

Lemma 8 *Any weak limit point of the measures $\mathbb{P}_N$ is concentrated on the set of solutions of the general frozen percolation equation* (12).

– *If $\mu(N) \equiv 1$, then the $\lambda(t)$ rate function of* (12) *is equal to the $\lambda(t)$ of* (4).
– *If $\mu(N) \ll 1$, then the $\lambda(t)$ rate function of* (12) *is equal to* 0.

Proof From (3) and (4) it follows that

$$\begin{aligned}
Lv_k^N(t) &:= \lim_{dt\to 0} \mathbf{E}\big(v_k^N(t+dt) - v_k^N(t) \,\big|\, \mathcal{F}_t\big) \\
&= \frac{1}{N}\frac{\mathcal{V}_k(\mathcal{V}_k - k)}{2}\left(-2\frac{k}{N}\right) + \left(\sum_{l\neq k}\frac{1}{N}\cdot\mathcal{V}_k\mathcal{V}_l\right)\left(-\frac{k}{N}\right) \\
&\quad + \left(\sum_{l=1}^{\lfloor\frac{k-1}{2}\rfloor}\frac{1}{N}\mathcal{V}_l\mathcal{V}_{k-l} + \mathbb{I}[2|k]\frac{1}{N}\frac{(\mathcal{V}_{\frac{k}{2}} - \frac{k}{2})\mathcal{V}_{\frac{k}{2}}}{2}\right)\frac{k}{N} - \lambda(t)\cdot\mu(N)\mathcal{V}_k\frac{k}{N} \\
&= -k\cdot((m_0(0) - \Phi^N(t)) + \lambda(t)\mu(N))\cdot v_k^N + \frac{k}{2}\sum_{l=1}^{k-1} v_l^N v_{k-l}^N \\
&\quad + \frac{1}{N}\left(k^2 v_k^N - \mathbb{I}[2|k]\cdot\frac{k^2}{4}v_{\frac{k}{2}}^N\right).
\end{aligned} \tag{71}$$

$M(t) = v_k^N(t) - v_k^N(0) - \int_0^t Lv_k^N(s)ds$ is a martingale and

$$\begin{aligned}
LM^2(t) &:= \lim_{dt\to 0}\mathbf{E}\big(M^2(t+dt) - M^2(t)\,\big|\,\mathcal{F}_t\big) = \lim_{dt\to 0}\mathbf{E}\big((v_k^N(t+dt) - v_k^N(t))^2\,\big|\,\mathcal{F}_t\big) \\
&\leq \left(2\frac{k}{N}\right)^2\cdot\left(\binom{\lfloor m_0(0)N\rfloor}{2}\frac{1}{N} + \lfloor m_0(0)N\rfloor\lambda(N)\right) = \mathcal{O}\left(\frac{k^2}{N}\right).
\end{aligned}$$

Thus $\mathbf{E}(M(T)^2) = \mathbf{E}(\int_0^t LM^2(s)ds) = \mathcal{O}(\frac{1}{N})$ if we fix k. It follows from Doob's maximal inequality that for all $\varepsilon > 0$, $k \geq 1$ and $T < +\infty$ we have

$$\lim_{N\to\infty}\mathbf{P}\left(\sup_{0\leq t\leq T}\left|v_k^N(t) - v_k^N(0) - \int_0^t Lv_k^N(s)ds\right| > \varepsilon\right) = 0. \tag{72}$$

If we rewrite this equation in terms of the functions $(w_k^N(\cdot))_{k=1}^\infty$ the claim of the lemma follows. □

Lemma 9 *If $\frac{1}{N} \ll \mu(N)$, $0 < \lambda_{inf} \leq \lambda(t)$ and $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$, then for any weak limit point $\mathbb{P}$ of the sequence of probability measures $\mathbb{P}_N$ on $\mathcal{W}_{\underline{\mathbf{w}}}[0,T]$ we have*

$$\mathbb{P}(\theta(t) \equiv 0) = 1. \tag{73}$$

The subcritical and critical parts of Theorem 2 follow from Lemma 8 and Lemma 9: any weak limit point $\mathbb{P}$ of the sequence $\mathbb{P}_N$ is concentrated on the set of frozen percolation evolutions satisfying (12) & (15). When $\mu(N) \equiv 1$, $\mathbb{P}$ is concentrated on the unique solution of (16), when $\frac{1}{N} \ll \mu(N) \ll 1$ then $\mathbb{P}$ is concentrated on the solution of (17).

In the rest of this section we discuss the proof of Lemma 9.

Lemma 10 *We consider a solution of the general frozen percolation equation* (12) *with initial condition* $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$. *If* $\lambda(t) \equiv 0$ *or* $0 < \lambda_{inf} \le \lambda(t) \le \lambda_{sup} < +\infty$ *then there is a constant* C^* *such that for all* $t_1 \le t_2$ *we have*

$$\theta(t_2) - \theta(t_1) \le C^* \cdot (t_2 - t_1). \tag{74}$$

Proof First we prove that there exists a constant C depending only on the initial data $\underline{\mathbf{v}}(0)$ and λ_{inf} such that

$$m_1(t) \le C. \tag{75}$$

If $V(t,x) = \sum_{k=1}^{\infty} v_k(t) e^{-kx}$ then by (12) we get

$$\dot{V}(t,x) = V'(t,x) \cdot ((m_0(0) - \Phi(t)) + \lambda(t) - V(t,x)), \tag{76}$$

$$\dot{V}'(t,x) = V''(t,x)\,(m_0(0) - \Phi(t) - \lambda(t) - V(t,x)) - V'(t,x)^2. \tag{77}$$

Substituting $V(t,x) - (m_0(0) - \Phi(t)) \le 0$ and $\frac{-V'(t,x)^3}{E_{sup}} \le V''(t,x)$ into (77) we get

$$\frac{d}{dt}\left(-V'(t,x)\right) \le V'(t,x)^2 \cdot \left(1 - \frac{\lambda_{inf}}{E_{sup}}\left(-V'(t,x)\right)\right)$$

which implies $-V'(t,x) \le \max\{m_1(0), \frac{E_{sup}}{\lambda_{inf}}\} =: C$ for all $x > 0$ and t by a "forbidden region"-argument. Thus by letting $x \to 0_+$ we get (75).

Now we show that for some constant C_2 we have

$$(V(t,x) - (m_0(0) - \Phi(t))\,V'(t,x) \le C_2 \tag{78}$$

for all $x > 0$. If $\lambda_{inf} \le \lambda(t)$, then by (75) and $-m_0(0) \le V(t,x) - (m_0(0) - \Phi(t)) \le 0$ we get (78) with $C_2 = m_0(0)C$.

Denote by $U(t,x) := V(t,x) - (m_0(0) - \Phi(t))$. If $\lambda(t) \equiv 0$ then by (76) and (77) we get

$$\begin{aligned}\frac{d}{dt}\left(U(t,x)V'(t,x)\right) &= -2V'(t,x)^2 U(t,x) - U(t,x)^2 V''(t,x) + V'(t,x)\frac{d}{dt}\Phi(t)\\ &\le (-U(t,x))V'(t,x)^2\left(2 - \frac{1}{E_{sup}}U(t,x)V'(t,x)\right).\end{aligned}$$

Thus we have (78) with $C_2 = \max\{m_1(0), 2E_{sup}\}$ again by a "forbidden region"-argument. Substituting the bounds (75) and (78) into (76) we get

$$\frac{d}{dt}\left(-V(t,x)\right) \le C_2 + C \cdot \lambda_{sup} =: C^*$$

for all x. Thus $V(t_1,x) - V(t_2,x) \le C^* \cdot (t_2 - t_1)$. Letting $x \to 0_+$ and substituting into (13) the claim of the lemma follows. □

We are going to prove Lemma 9 by contradiction: in Lemma 11 we show that if $\theta(\cdot) \not\equiv 0$ in the limit, then there is a positive time interval such that $\theta(t)$ has a positive lower bound, and that this implies that even in the convergent sequence of finite-volume models, a lot of mass is contained in arbitrarily big components on this interval. Than in subsequent lemmas we prove that these big components indeed burn, which produces such a big increase in the value of the burnt mass $\Phi(\cdot)$ that is in contradiction with $\Phi(\cdot) \le m_0(0)$.

For any frozen percolation evolution obtained from a frozen percolation Markov process on a finite number of vertices we obviously have $\theta^N(t) \equiv 0$ (see (9) and (13)), thus

$$\forall K \in \mathbb{N} \quad \sum_{k>K} v_k^N(t) = m_0(t) - w_K^N(t) = m_0(0) - \Phi^N(t) - w_K^N(t). \tag{79}$$

Lemma 11 *If $\mathbb{P}_N \Rightarrow \mathbb{P}$ where $\mathbb{P}$ does not satisfy (73) on $[0,T]$, then there exist ε_1, ε_2, $\varepsilon_3 > 0$ and a deterministic $t^* \in [\varepsilon_1, T]$ such that for every $K < +\infty$, every $m < +\infty$ and every sequence*

$$t^* - \varepsilon_1 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_m < \beta_m < t^*$$

there exists an $N_0 < +\infty$ such that for every $N \geq N_0$ and $1 \leq i \leq m$ we have

$$\mathbb{P}_N \left(\max_{\alpha_i \leq t \leq \beta_i} \sum_{k>K} v_k^N(t) > \varepsilon_2 \right) > \varepsilon_3. \tag{80}$$

Proof First we prove that if $\mathbb{P}$ does not satisfy (73) then there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and $\varepsilon_1 \leq t^* \leq T$ such that

$$\mathbb{P}\Big(\inf_{t^*-\varepsilon_1 \leq t \leq t^*} \theta(t) > \varepsilon_2 \Big) > \varepsilon_3. \tag{81}$$

Since (73) is violated, we have $\mathbb{P}(\sup_{0 \leq t \leq T} \theta(t) > \varepsilon) > \varepsilon$ for some $\varepsilon > 0$.

Let $L := \lfloor \frac{2C^*T}{\varepsilon} \rfloor$ and $t_i := \frac{\varepsilon i}{2C^*}$ for $1 \leq i \leq L$ where C^* is the constant in (74).

By Lemma 8 the random frozen percolation evolution obtained as a weak limit point satisfies (12) with a possibly random control function Φ, so (74) holds $\mathbb{P}$-almost surely for the random element of $\mathcal{W}_{\underline{\mathbf{w}}}[0,T]$ obtained as a weak limit point.

Since $\theta(0) = 0$ we have

$$\Big\{ \sup_{0 \leq t \leq T} \theta(t) > \varepsilon \Big\} \subseteq \bigcup_{i=1}^{L} \Big\{ \theta(t_i) > \frac{\varepsilon}{2} \Big\}$$

almost surely with respect to $\mathbb{P}$. Thus $\mathbb{P}(\theta(t^*) > \frac{\varepsilon}{2}) > \frac{\varepsilon}{L}$ for some $t^* \in \{t_1, \dots t_L\}$. Using (74) again (81) follows with $\varepsilon_1 := \frac{\varepsilon}{4C^*}$, $\varepsilon_2 := \frac{\varepsilon}{4}$, $\varepsilon_3 = \frac{\varepsilon}{L}$.

Now given K and the intervals $[\alpha_i, \beta_i]$, $1 \leq i \leq m$ we define the continuous functionals $f_i : \mathcal{W}_{\underline{\mathbf{w}}}[0,T] \to \mathbb{R}$ by

$$f_i\left((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot) \right) := \frac{1}{\beta_i - \alpha_i} \int_{\alpha_i}^{\beta_i} \big(m_0(0) - w_K(t) - \Phi(t) \big) dt.$$

Thus for all i

$$H_i := \{ \left((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot) \right) \in \mathcal{W}_{\underline{\mathbf{w}}}[0,T] : f_i\left((w_k(\cdot))_{k=1}^{\infty}, \Phi(\cdot) \right) > \varepsilon_2 \}$$

is an open subset of $\mathcal{W}_{\underline{\mathbf{w}}}[0,T]$ with respect to the topology of Definition 7. Thus by the definition of weak convergence of probability measures we have

$$\lim_{N \to \infty} \mathbb{P}_N(H_i) \geq \mathbb{P}(H_i) \geq \mathbb{P}\left(\inf_{t^*-\varepsilon_1 \leq t \leq t^*} \theta(t) > \varepsilon_2 \right) > \varepsilon_3$$

from which the claim of the lemma easily follows by (79). □

Lemma 12 *If* $\frac{1}{N} \ll \mu(N)$ *and* $0 < \lambda_{inf} \le \lambda(t)$, *then for every* $\varepsilon_2 > 0$ *there is a* $\varepsilon_4 > 0$ *such that for every* $\tilde{t} > 0$ *there is a* K *and an* N_1 *such that*

$$\forall N \ge N_1 \quad \sum_{k>K} v_k^N(0) \ge \varepsilon_2 \implies \mathbb{E}_N\left(\Phi^N(\tilde{t})\right) \ge \varepsilon_4. \tag{82}$$

The proof of Lemma 12. will follow as a consequence of Lemmas 13 and 14.

Proof of Lemma 9 We are going to show that if there is a sequence $\mathbb{P}_N$ such that the weak limit point $\mathbb{P}$ violates (73) then for some N we have

$$\mathbb{E}_N\left(\Phi^N(T)\right) > m_0(0) \tag{83}$$

which is in contradiction with (13).

We define ε_1, ε_2, $\varepsilon_3 > 0$ and t^* using Lemma 11. Next, we define ε_4 using this ε_2 and Lemma 12. Given these, we choose $\tilde{t}$ be so small that

$$\left\lfloor \frac{\varepsilon_1}{2\tilde{t}} \right\rfloor \varepsilon_3 \varepsilon_4 > m_0(0).$$

We choose K and N_1 big enough so that (82) holds for this $\tilde{t}$. Further on, we fix the intervals $[\alpha_i, \beta_i]$, $1 \le i \le m = \lfloor \frac{\varepsilon_1}{2\tilde{t}} \rfloor$ so that $\alpha_{i+1} - \beta_i > \tilde{t}$ holds for all i and also $T - \beta_m > \tilde{t}$ holds. We choose N_0 such that (80) holds and let $N := \max\{N_0, N_1\}$.

Finally, we define the stopping times $\tau_1, \tau_2, \ldots, \tau_m$ by

$$\tau_i := \beta_i \wedge \min\left\{ t : t \ge \alpha_i \text{ and } \sum_{k>K} v_k^N(t) \ge \varepsilon_2 \right\}.$$

We have $\tau_i + t^* \le \beta_i + t^* < \alpha_{i+1} \le \tau_{i+1}$.

Using the strong Markov property, (82) and (80), the inequality (83) follows:

$$\begin{aligned} \mathbf{E}\left(\Phi^N(T)\right) &\ge \sum_{i=1}^m \mathbf{E}\left(\Phi^N(\tau_i + t^*) - \Phi^N(\tau_i)\right) \\ &\ge \sum_{i=1}^m \mathbf{E}\left(\mathbf{E}\left((\Phi^N(\tau_i + t^*) - \Phi^N(\tau_i)) \mathbb{I}\left[\sum_{k>K} v_k^N(\tau_i) \ge \varepsilon_2 \right] \middle| \mathcal{F}_{\tau_i} \right) \right) \\ &\ge \sum_{i=1}^m \varepsilon_4 \mathbf{P}\left(\sum_{k>K} v_k^N(\tau_i) \ge \varepsilon_2 \right) \ge m\varepsilon_4\varepsilon_3 > m_0(0). \end{aligned}$$

□

For a frozen percolation evolution defined by (9) we have

$$U(t,x) = \sum_{k\ge1} v_k^N(t) e^{-kx} - (m_0(0) - \Phi^N(t)) = V(t,x) - m_0^N(t) = \sum_{k\ge1} v_k^N(t)\left(e^{-kx} - 1\right). \tag{84}$$

We will make use of the following generating function estimates in the proof of Lemma 13.

If $U(x) = \sum_{k\ge1} v_k(e^{-kx} - 1)$ where $\underline{\mathbf{v}} \in \mathbf{V}$ then

$$\sum_{k>K} v_k \ge \varepsilon \implies U(1/K) \le (e^{-1} - 1)\varepsilon, \tag{85}$$

$$U(1/K) \le -\varepsilon \implies \sum_{k > \frac{\varepsilon K}{2}} v_k \ge \varepsilon/2. \tag{86}$$

Lemma 13 *There are constants* $C_1 < +\infty$, $C_2 > 0$, $C_3 > 0$ *such that if*

$$\sum_{k>K} v_k^N(0) \ge \varepsilon_2 \tag{87}$$

for all N *then*

$$\lim_{N\to\infty} \mathbf{P}\left(\sum_{k > C_3 \varepsilon_2 N^{1/3}} v_k^N(\bar{t}) + \Phi^N(\bar{t}) \ge C_2 \varepsilon_2 \right) = 1, \tag{88}$$

where $\bar{t} = \frac{C_1}{K\varepsilon_2}$.

Sketch proof If we let $N \to \infty$ immediately, then by Lemma 8 we get that the limiting functions $v_1(t), v_2(t), \ldots$ solve (14) with initial condition $\underline{\mathbf{v}}(0)$, a possibly random control function $\Phi(t)$ and some nonnegative rate function $\lambda(t)$.

The $N \to \infty$ limit of (88) is

$$\theta(\bar{t}) + \Phi(\bar{t}) \ge C_2 \varepsilon_2. \tag{89}$$

Now we prove that if $\underline{\mathbf{v}}(\cdot)$ is a solution of (14) then $\sum_{k>K} v_k(0) \ge \varepsilon_2$ implies (89) with $C_1 = 4$ and $C_2 = \frac{1}{4}$. This proof will also serve as an outline of the proof of Lemma 13.

In order to prove (89) define $V(t,x)$ by (39). Thus $V(t,x)$ solves

$$\dot{V}(t,x) = V'(t,x) \cdot (m_0(0) - \Phi(t) + \lambda(t) - V(t,x)). \tag{90}$$

Define $U(t,x)$ by (41). Define the characteristic curve $x(\cdot)$ by

$$\dot{x}(t) = V(t, x(t)) - (m_0(0) - \Phi(t) + \lambda(t)), \qquad x(0) = \frac{1}{K}. \tag{91}$$

Let $\nu(t) := V(t, x(t))$. Now by (90) and (91) we get

$$\dot{\nu}(t) = \dot{V}(t, x(t)) + V'(t, x(t))\dot{x}(t) = 0. \tag{92}$$

Thus $\nu(t) \equiv \nu(0)$, moreover by (41) we get $U(t,x(t)) - U(0,x(0)) = \Phi(t)$ and by $V(t,x(t)) \equiv V(0,x(0))$, $V(0,x(0)) - m_0(0) = U(0,x(0))$ and (91) we get

$$x(t) = \frac{1}{K} + \int_0^t \Phi(s)\,ds - \int_0^t \lambda(s)\,ds + t \cdot U\left(0, \frac{1}{K}\right). \tag{93}$$

By (85) we have $U(0, \frac{1}{K}) \le -\frac{1}{2}\varepsilon_2$. In order to prove that $\theta(\bar{t}) + \Phi(\bar{t}) \ge \frac{1}{4}\varepsilon_2$ with $\bar{t} = \frac{4}{K\varepsilon_2}$ we consider two cases:

If $\Phi(\bar{t}) \ge \frac{1}{4}\varepsilon_2$ then we are done. If $\Phi(\bar{t}) < \frac{1}{4}\varepsilon_2$ define $\tau := \min\{t : x(t) = 0\}$. By (93) we have

$$x(\bar{t}) \le \frac{1}{K} + \bar{t} \cdot \Phi(\bar{t}) + \bar{t} \cdot \left(-\frac{1}{2}\varepsilon_2\right) < \frac{1}{K} + \frac{1}{K} - \frac{2}{K} = 0.$$

Thus $\tau \leq \bar{t}$.

$$-\theta(\tau) = U(\tau, 0) = U(\tau, x(\tau)) = U\left(0, \frac{1}{K}\right) + \Phi(\tau) \leq -\frac{1}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_2 = -\frac{1}{4}\varepsilon_2.$$

Thus $\frac{1}{4}\varepsilon_2 \leq \theta(\tau) \leq \theta(\tau) + \Phi(\tau) \leq \theta(\bar{t}) + \Phi(\bar{t})$ because by (13) the function $\theta(t) + \Phi(t)$ is increasing. □

To make this proof work for Lemma 13 we have to deal with the fluctuations caused by randomness and combinatorial error terms.

Proof Given a frozen percolation evolution obtained from a Markov process by (9) define U and V by (84).

Using (71) a straightforward calculation shows that

$$\begin{aligned} LV(t,x) &:= \lim_{h \to 0_+} \frac{1}{h} \mathbf{E}\big(V(t+h,x) - V(t,x) \,\big|\, \mathcal{F}_t\big) \\ &= V'(t,x)\left((m_0(0) - \Phi^N(t)) + \lambda(t)\mu(N) - V(t,x)\right) \\ &\quad + \frac{1}{N}\left(V''(t,x) - V''(t,2x)\right). \end{aligned} \tag{94}$$

Given the random function $V(t,x)$ we define the random characteristic curve $x(t)$ similarly to (91):

$$\dot{x}(t) = V(t,x(t)) - \left((m_0(0) - \Phi^N(t)) + \lambda(t)\mu(N)\right), \qquad x(0) = \frac{1}{K}. \tag{95}$$

This ODE is well-defined although $V(t,x)$ is not continuous in t, but almost surely it is a step function with finitely many steps which is a sufficient condition to have well-posedness for the solution of (95). Define $\nu(t) := V(t,x(t))$.

$$x(t) = \frac{1}{K} + \int_0^t (\nu(s) - \nu(0))\,ds + \int_0^t \Phi^N(s)ds - \mu(N)\int_0^t \lambda(s)ds + t \cdot U\left(0, \frac{1}{K}\right). \tag{96}$$

Putting together (94) and (95) we get

$$\lim_{h \to 0_+} \frac{1}{h}\mathbf{E}\big(\nu(t+h) - \nu(t) \,\big|\, \mathcal{F}_t\big) = \frac{1}{N}\left(V''(t,x(t)) - V''(t,2x(t))\right) = \mathcal{O}\left(\frac{1}{N}V''(t,x(t))\right). \tag{97}$$

Thus $\widetilde{\nu}(t) = \nu(t) - \int_0^t \frac{1}{N}(V''(s,x(s)) - V''(s,2x(s)))ds$ is a martingale and by (4) and (3) we get

$$\begin{aligned} &\lim_{h \to 0_+} \frac{1}{h}\mathbf{E}\big(\widetilde{\nu}(t+h)^2 - \widetilde{\nu}(t)^2 \,\big|\, \mathcal{F}_t\big) \\ &\quad = \lim_{h \to 0_+} \frac{1}{h}\mathbf{E}\Big(\big(V(t+h,x(t)) - V(t,x(t))\big)^2 \,\Big|\, \mathcal{F}_t\Big) \\ &\quad \leq \frac{1}{2}\sum_{k,l=1}^{N}\left(\frac{k+l}{N}e^{-(k+l)x(t)} - \frac{k}{N}e^{-kx(t)} - \frac{l}{N}e^{-lx(t)}\right)^2 v_k^N(t)v_l^N(t)N \end{aligned}$$

$$+\sum_{l=1}^{N}\left(\frac{l}{N}e^{-lx(t)}\right)^2\mu(N)\lambda(t)v_l^N(t)N=\mathcal{O}\left(\frac{1}{N}V''(t,x(t))\right). \tag{98}$$

Define the stopping time

$$\tau_N := \min\{t : x(t) = N^{-1/3}\}.$$

(Note that we could replace $N^{-1/3}$ by $N^{-\gamma}$, $0<\gamma<1/2$ without changing the proof.)

It follows from (46), (97), (98) and Doob's maximal inequality that

$$\sup_{0\le t\le T}|\nu(t\wedge\tau_N\wedge T)-\nu(0)|\Rightarrow 0 \quad \text{as } N\to\infty. \tag{99}$$

By (85) and (87) we have

$$U(0,x(0))\le(e^{-1}-1)\varepsilon_2=:-\varepsilon_5. \tag{100}$$

Let

$$A_N := \left\{\int_0^{\tau_N\wedge T}|\nu(s)-\nu(0)|\,ds\le\frac{1}{K}\right\}\cap\left\{|\nu(\tau_N\wedge T)-\nu(0)|\le\varepsilon_5/3\right\},$$

$$B_N := \left\{\Phi^N(\tau_N)\le\varepsilon_5/3\right\}.$$

$$\bar{t} := \frac{3}{K\,|U(0,x(0))|}\le\frac{3}{K\varepsilon_5},$$

We are going to show that there are constants $C_2, C_3<+\infty$ such that

$$A_N\subseteq\left\{\sum_{k>C_3\varepsilon_2N^{1/3}}v_k^N(\bar{t})+\Phi^N\left(\bar{t}\right)\ge C_2\varepsilon_2\right\} \tag{101}$$

which together with (99) implies $\lim_{N\to\infty}\mathbf{P}(A_N)=1$ and (88).

First we show that

$$A_N\cap B_N\subseteq\{\tau_N\le\bar{t}\}. \tag{102}$$

If we assume indirectly that A_N, B_N and $\tau_N>\bar{t}$ hold then $\int_0^{\bar{t}}|\nu(s)-\nu(0)|ds\le\frac{1}{K}$, so by (96) we get

$$x(\bar{t})\le\frac{1}{K}+\frac{1}{K}+\int_0^{\bar{t}}\Phi^N(s)ds+\bar{t}U(0,x(0))\le-\frac{1}{K}+\bar{t}\frac{\varepsilon_5}{3}\le 0.$$

But $x(\bar{t})\le 0$ is in contradiction with $\tau_N>\bar{t}$, thus (102) holds. Assuming A_N and B_N we obtain

$$|U(\tau_N,x(\tau_N))-U(0,x(0))|\le|\nu(\tau_N)-\nu(0)|+\Phi^N(\tau_N)\le\varepsilon_5/3+\varepsilon_5/3$$

which together with (100) implies $A_N\cap B_N\subseteq\{U(\tau_N,N^{-1/3})\le-\varepsilon_5/3\}$.

By (86)

$$A_N \subseteq (A_N \cap B_N) \cup B_N^c \subseteq \left\{ \sum_{k > N^{1/3}\varepsilon_5/6} v_k^N(\tau_N) \geq \varepsilon_5/6 \right\} \cup \left\{ \Phi^N(\tau_N) > \varepsilon_5/3 \right\}$$

$$\subseteq \left\{ \sum_{k > C_3 \varepsilon_2 N^{1/3}} v_k^N(\tau_N) + \Phi^N(\tau_N) \geq C_2\varepsilon_2 \right\}$$

with $C_2 = C_3 = (1 - e^{-1})/6$. But $\sum_{k > C_3\varepsilon_2 N^{1/3}} v_k^N(t) + \Phi^N(t)$ is a monotone increasing function of t, from which (101) follows. □

Lemma 14 *There are constants $C_4 < +\infty$, $C_5 > 0$ such that if*

$$\sum_{k > C_3 \varepsilon_2 N^{1/3}} v_k^N(0) \geq C_2\varepsilon_2/2$$

for all N then with

$$\bar{t}_N := C_4 \varepsilon_2^{-2} \left(N^{-1/3} \log(N) + (N\mu(N))^{-1} \right) \tag{103}$$

we have

$$\lim_{N\to\infty} \mathbf{E}\left(\Phi^N(\bar{t}_N)\right) \geq C_5\varepsilon_2. \tag{104}$$

Remark 5 The upper bound (103) is technical: on one hand it is not optimal, on the other hand, for the proof of Lemma 12 we only need $\bar{t}_N \ll 1$ as $N \to \infty$.

Proof If v is a vertex of the graph $G(N,t)$ let $\mathcal{C}_N(v,t)$ denote the connected component of v at time t. Denote by $\tau_b(v)$ the freezing/burning time of v.

$$\mathcal{H}_N(t) := \{ v : |\mathcal{C}_N(v,0)| \geq C_3 \varepsilon_2 N^{\frac{1}{3}} \text{ and } \tau_b(v) > t \}.$$

We fix a vertex $v \in \mathcal{H}_N(0)$.

$$c_N(t) := \frac{1}{N} |\mathcal{C}_N(v,t)|,$$

$$w_N(t) := \frac{1}{N} |\mathcal{H}_N(t)|,$$

$$z_N(t) := \frac{1}{N} |\mathcal{H}_N(0) \setminus \mathcal{H}_N(t)| = w_N(0) - w_N(t).$$

Thus $c_N(t)$ is an increasing process until $\tau_b(v)$, $w_N(t)$ is decreasing, $z_N(t)$ is increasing. We consider the right-continuous version of the processes $c_N(t)$, $w_N(t)$, $z_N(t)$.

$$w_N(0) \geq C_2\varepsilon_2/2 =: \varepsilon_6.$$

We are going to prove that there are constants $C_4 < +\infty$, $C_5 > 0$ such that

$$\lim_{N\to\infty} \mathbf{E}\left(z_N(\bar{t}_N)\right) \geq C_5\varepsilon_2 \tag{105}$$

with $\bar{t}_N$ defined as in (103). This implies (104).

We define the stopping times

$$\tau_w := \min\{t : w_N(t) < \varepsilon_6/2\},$$
$$\tau_g := \min\{t : c_N(t) > \varepsilon_6/4\},$$
$$\tau := \tau_b(v) \wedge \tau_w \wedge \tau_g.$$

Let $\bar{N} := C_3 \varepsilon_2 N^{\frac{1}{3}}$. Since $v \in \mathcal{H}_n(0)$ we have

$$c_N(t) \geq c_N(0) = \frac{|\mathcal{C}_N(v,0)|}{N} \geq \frac{\bar{N}}{N}.$$

If $\mathcal{C}_N(v,t)$ is connected to a vertex in $\mathcal{H}_N(t)$ by a new edge at time t then

$$\begin{aligned}
&c_N(t_+) - c_N(t_-) \geq \frac{\bar{N}}{N},\\
&\log(c_N(t_+)) - \log(c_N(t_-)) \geq \log\left(1 + \frac{\bar{N}}{N c_N(t_-)}\right) \geq \frac{\log(2)\bar{N}}{N c_N(t_-)},\\
&\lim_{dt\to 0} \frac{1}{dt} \mathbf{E}\big(\log(c_N(t+dt)) - \log(c_N(t)) \,\big|\, \mathcal{F}_t\big)\\
&\quad \geq \frac{\log(2)\bar{N}}{N c_N(t)} \lim_{dt\to 0} \frac{1}{dt} \mathbf{P}\left(c_N(t+dt) - c_N(t) \geq \frac{\bar{N}}{N} \,\middle|\, \mathcal{F}_t\right)\\
&\quad \geq \frac{\log(2)\bar{N}}{N c_N(t)} \cdot \frac{1}{N} \left|\mathcal{C}_N(v,t)\right| \left(|\mathcal{H}_N(t)| - |\mathcal{C}_N(v,t)|\right) \mathbb{I}_{\{t\ \leq \tau_b(v)\}}\\
&\quad \geq \log(2)\bar{N} \cdot (w_N(t) - c_N(t))\, \mathbb{I}_{\{t\ \leq \tau_b(v)\}}\\
&\quad \geq \log(2)\bar{N} \frac{\varepsilon_6}{4} \mathbb{I}_{\{t \leq \tau\}} = N^{1/3} \frac{\log(2)}{8} \cdot C_2 \cdot C_3 \cdot (\varepsilon_2)^2 \cdot \mathbb{I}_{\{t \leq \tau\}} =: a \cdot \mathbb{I}_{\{t \leq \tau\}}.
\end{aligned}$$

Thus $\log(c_N(t)) - a \cdot (t \wedge \tau)$ is a submartingale. Using the optional sampling theorem we get

$$\log(m_0(0)) - a \cdot \mathbf{E}(\tau) \geq \mathbf{E}\left(\log(c_N(\tau))\right) - a \cdot \mathbf{E}(\tau) \geq \log(c_N(0)) \geq -\log(N).$$

By Markov's inequality we obtain that for some constant $C < +\infty$

$$\mathbf{P}\left(\tau \leq C N^{-1/3} \varepsilon_2^{-2} \log(N)\right) \geq \frac{1}{2}$$

if N is sufficiently large.

If $\tau_g \leq \tau_b(v)$, then $\mathcal{C}_N(v,\tau_g) > \frac{\varepsilon_6}{4} N$, so $\mathbf{E}(\tau_b(v) - \tau_g) \leq (N\mu(N)\lambda_{inf})^{-1} \frac{4}{\varepsilon_6}$, which implies

$$\mathbf{P}\left(\tau_w \wedge \tau_b(v) \leq C N^{-1/3} \varepsilon_2^{-2} \log(N) + C'(N\mu(N))^{-1} \varepsilon_2^{-1}\right) \geq \frac{1}{4}$$

for some constant C'. Define $\bar{t}$ of (103) with $C_4 := \max\{C, C'\}$. Using the linearity of expectation we get

$$\mathbf{E}\left(z(\bar{t})\right) = \mathbf{E}\left(\frac{1}{N}\sum_{w\in\mathcal{H}_N(0)} \mathbb{I}_{\{\tau_b(w)\le\bar{t}\}}\right) \ge \varepsilon_6 \mathbf{P}\left(\tau_b(v)\le\bar{t}\right).$$

The inequality $\mathbb{I}_{\{\tau_w\le\bar{t}\}}\frac{\varepsilon_6}{2} \le z(\bar{t})$ follows from the definition of τ_w.

$$\frac{1}{4} \le \mathbf{P}\left(\tau_w \wedge \tau_b(v) \le \bar{t}\right) \le \mathbf{P}\left(\tau_w \le \bar{t}\right) + \mathbf{P}\left(\tau_b(v)\le\bar{t}\right) \le \mathbf{E}\left(z(\bar{t})\right)\frac{2}{\varepsilon_6} + \mathbf{E}\left(z(\bar{t})\right)\frac{1}{\varepsilon_6}.$$

From which (105) follows. □

Lemma 12 is a straightforward consequence of Lemma 13 and Lemma 14.

5 Properties of the Solutions of the Frozen Percolation Equations

Proof of Theorem 5 It is clear from (63) and (65) that $\varphi(t)$ is continuous. In order to prove (23) we need Example (c) of Theorem 4 of Chap. XIII.5 of [4]. By (55)

$$X''(t,0) = \frac{1}{E(t,0)} = \frac{1}{E(0,t)} = \frac{1}{\varphi(t)},$$

$$X(t,u) = \frac{1}{2\varphi(t)}u^2 + \mathcal{O}(u^3), \qquad \lim_{x\to 0}\frac{-U(t,x)}{\sqrt{x}} = \sqrt{2\varphi(t)}.$$

By the Tauberian theorem for any $t \ge T^g$ each of the relations

$$-U(t,x) \sim x^{1-1/2}\sqrt{2\varphi(t)} \quad \text{and} \quad \sum_{k=K}^{\infty} v_k(t) \sim \frac{1}{\Gamma(\frac{1}{2})}K^{1/2-1}\sqrt{2\varphi(t)}$$

implies the other, that is for any $t \ge T^g$

$$\lim_{x\to 0}\frac{-U(t,x)}{\sqrt{x}} = \sqrt{2\varphi(t)} \quad \Longleftrightarrow \quad \lim_{K\to\infty} K^{\frac{1}{2}}\sum_{k=K}^{\infty} v_k(t) = \sqrt{\frac{2\varphi(t)}{\pi}}.$$

□

In order to compare the solutions of (19) and (17) we apply the transformations

$$\underline{\mathbf{v}}(t) \to U(t,x) \to X(t,u) \to G(t,w) \tag{106}$$

to the solutions of the alternating equations:

The integral equation

$$U(t,x) = U(0,x) + \int_0^t -U(s,x)U'(s,x)ds + \Phi(t) \tag{107}$$

holds, but $\Phi(t)$ is constant between burning times and jumps by $\theta(T_i^b)$ at T_i^b, which means that the giant component is burnt:

$$\lim_{\varepsilon\to 0} -U(T_i^b+\varepsilon, 0) = \lim_{\varepsilon\to 0} \theta(T_i^b+\varepsilon) = \theta(T_{i+}^b) = 0.$$

By Lemma 1 the formulae (45), (53), (54) and (55) are valid (with rate function $\lambda(t) \equiv 0$).

In between the burning times $T_i^b < t \le T_{i+1}^b$ we have

$$X(t,u) = X(T_{i+}^b, u) + (t - T_i^b)u \quad \text{and} \quad G(t,w) = G(T_{i+}^b, w + (t - T_i^b)).$$

If $t - T_i^b > w^*(T_{i+}^b)$ then $\underline{\mathbf{v}}(t)$ is supercritical:

$$X'(t,0) > 0, \qquad \theta(t) > 0, \qquad X(t, -\theta(t)) = 0, \qquad X'(t, -\theta(t)) < 0.$$

$\min_w G(t,w) = 0$ still holds, but $\operatorname{argmin}_w G(t,w) = w^*(t) < 0$ in the supercritical phase. Thus $-X'(t,0) = w^*(t)$ is well-defined for all $t \ge 0$ for the solutions of (16), (17) and (19) as well, moreover (59) holds. For the solutions of (19) $w^*(t)$ is left-continuous.

By $G(t, w^*(t)) \equiv 0$, (53) and (70) we get

$$\int_0^t \Phi(s)ds = G(0, t + w^*(t)) - w^*(t)F(0, t + w^*(t)) \tag{108}$$

for the solutions of (19).

If $\underline{\mathbf{v}}(t)$ is the solution of (16), (17) or (19) started from $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$, then (55) holds: the evolution of the critical core does not depend on the rate of lightnings. One extra parameter is needed to determine $\underline{\mathbf{v}}(t)$ and $\theta(t)$: if we know $w^*(t)$, then

$$F(t,w) = \int_{w^*(t)}^{w} E(0, t+y)dy \quad \text{and} \quad G(t,w) = \int_{w^*(t)}^{w} (w-y)E(0, t+y)dy \tag{109}$$

has all the information about $\underline{\mathbf{v}}(t)$ and $\theta(t)$, since the transformations (106) are invertible (using analytic extensions).

Proof of Claim 1 First assume $m_0(0) = 1$. As a consequence of Remark 4 we can see that

$$E(t,w) = E(0, t+w) = \frac{1}{(w+t)^2} = \frac{1}{t^2} E\left(1, \frac{w}{t}\right), \tag{110}$$

but this is the critical core of $\frac{1}{t}\underline{\mathbf{v}}(1)$, and together with $w^*(t) \equiv 0$ for $t \ge T^g = 1$ the identity $v_k(t) = \frac{1}{t} v_k(1)$ follows. We get the explicit formula for $v_k(1)$ in the following way: since $X(1,u) = X(0,u) + u$, the inverse function of $V(1,x)$ is $-\log(v) + v - 1$, thus

$$V(1,x) = -W\left(-e^{-(x+1)}\right) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} e^{-k} e^{-kx},$$

where W is the Lambert W function, the inverse function of $z \mapsto ze^z$.

If $m_0(0) \ne 1$ but we still have a monodisperse initial condition then (110) still holds and for $t \ge \frac{1}{m_0(0)} = T^g$ we have $w^*(t) = 0$ thus $v_k(t) = \frac{1}{t} \frac{k^{k-1}}{k!} e^{-k}$ must hold. □

Proof of Theorem 7 Let $H(w) := F(0,w) - m_0(0)$, thus $H(-\frac{1}{V'(0,x)}) = -V(0,x)$ by (52). Using Lemma 5 and (54) we get

$$F(t,w) = H(t+w) - H(t) \quad \text{and} \quad m_0(t) = F(t,+\infty) = -H(t)$$

for $t \geq T^g$. $v_1(0) > 0$ implies $\lim_{x\to\infty} -\frac{V'(0,x)}{V(0,x)} = 1$, so $\lim_{t\to\infty} t \cdot H(tw) = -\frac{1}{w}$, from which $\lim_{t\to\infty} t m_0(t) = 1$ follows. Moreover

$$1 - \frac{1}{w+1} = \lim_{t\to\infty} t \cdot (H(t\cdot(w+1)) - H(t)) = \lim_{t\to\infty} t \cdot F(t,tw) = \lim_{t\to\infty} \hat{F}(t,w),$$

where $\hat{v}_k(t) = t v_k(t)$. This implies the pointwise convergence of the monotone functions $\hat{X}'(t,u)$, $\hat{X}(t,u)$, $\hat{U}(t,x)$ and $\hat{V}(t,x)$ to the desired limit as $t \to \infty$. The convergence of $\hat{v}_k(t)$ to $\frac{k^{k-1}}{k!}e^{-k}$ follows from the continuity theorem of Laplace transforms. □

Proof of Theorem 6 It is easy to check that if $\tilde{v}_k(t) = v_k(t)e^{-kx^*(t)}$, then $\tilde{V}(t,x) = V(t,x+x^*(t))$, so $\tilde{x}^*(t) = 0$ and $\tilde{w}^*(t) = 0$, but $\tilde{E}(t,w) = E(0,t+w) = E(t,w)$, so $\underline{\tilde{\mathbf{v}}}(t)$ is identical to the solution of (17) at time t. □

Proof of Theorem 10 If we consider the solution of (16) with given initial data and lightning rate function $\lambda(t) \geq 0$, $0 \leq t \leq T$ then (53) provides us with a relation between our cost $(\int_0^T \lambda(t)dt)$ and reward $(\int_0^T \Phi(t)dt)$.

We prove (35) by considering the cases $T \geq T^g$ and $T \leq T^g$ separately.

According to (62), for $T \geq T^g$ we get

$$0 \leq G^{sub}(T,0) = \int_0^T \Phi^{crit}(t)dt - \int_0^T \Phi^{sub}(t)dt + \int_0^T \lambda(t)dt$$

by substituting $w = 0$ into (53).

For $T \leq T^g$, we want to prove $0 \geq \int_0^T \Phi^{sub}(t)dt - \int_0^T \lambda(t)dt$. Substitute $w = T^g - T$ into (53). Since $G(0,T^g) = 0$ and $(T^g - T)\Phi^{sub}(T) \geq 0$ we get

$$0 \leq G^{sub}(T, T^g - T) \leq -\int_0^T \Phi^{sub}(t)dt + \int_0^T \lambda(t)dt.$$

The proof of the extremum property (36) is equally simple. □

If we want to maximize our cost functional for a fixed $T > T^g$, the optimal control is not unique, since the only thing we need for

$$\int_0^T \Phi^{sub}(t)dt - \int_0^T \lambda(t)dt = \int_0^T \Phi^{crit}(t)dt \tag{111}$$

to hold is $G^{sub}(T,0) = 0$: if $\underline{\mathbf{v}}(T)$ is critical at time T, then the value of the functional is optimal.

Proof of Remark 2 In order to prove (37) first pick an arbitrary $\lambda > 0$ and solve (67) with constant $\lambda(t) = \lambda$. Since $w^*(t) > 0$ and $w^*(0) = T^g$ there is a $0 < t^* \leq T$ such that $w^*(t^*) = T - t^*$, and the lightning rate function $\lambda(t) = \lambda \cdot \mathbb{I}[t \leq t^*]$ makes T a critical time, so (111) holds, thus (37).

Now we prove (38). By using (108) we have to show that

$$G(0, T + w^*(T)) - (w^*(T) - \varepsilon)F(0, T + w^*(T)) > G(0, T) + \varepsilon F(0, T).$$

Using $G(0, T + w^*(T)) - G(0, T) > w^*(T)F(0, T)$ it is easy to see that $0 < w^*(T) \le \varepsilon$ is sufficient for this to hold. If there is a $T^g < t^* \le T$ such that $-X'(t^*, -\theta(t^*)) = T - t^* + \varepsilon$, then burning the giant component at time t^* we get $-X'(t^*_+, 0) = T - t^* + \varepsilon$ and $-X'(T, 0) = w^*(T) = \varepsilon$. If not, then burning at time T yields $0 < -X'(T, -\theta(T)) = w^*(T_+) < \varepsilon$. □

6 Proof of the Subcritical Limit Theorem

In order to prove Theorem 8, we need to know more about the solution of (67).

Lemma 15 *If $y(t)$ is the solution of the differential equation $\dot{y}(t) = \frac{c}{y(t)} - 1$ with initial condition $y(0) = T^g$ and $t \ge T^g + c\log(\frac{T^g}{c})$ then $y(t) \le 2c$.*

Proof The solution of this differential equation is

$$y(t) = c \cdot \left(1 + W\left(\exp\left(\frac{T^g - t}{c} - 1\right) \cdot \left(\frac{T^g}{c} - 1\right)\right)\right), \tag{112}$$

where W is the Lambert W function. Thus $W(x) \le x$ and our claim follows. □

Lemma 16 *If $w^*(t)$ is the solution of (67) with constant $\lambda(t) \equiv \lambda \le 1$ then there exist d_1 and d_2 which depend only on $\underline{\mathbf{v}}(0)$ and T such that*

$$T^g + d_1\lambda\log\left(\frac{1}{\lambda}\right) \le t \le T \quad \Longrightarrow \quad \left|w^*(t) - \frac{\lambda}{E(0,t)}\right| \le d_2\lambda^2.$$

Proof We have a uniform a priori bound $w^*(t) \le w^*_{max}$ for all $\lambda \le 1$ depending only on the initial data and T by (69). Thus by Lemma 6 we have

$$0 < E_{inf} := E_{inf}(T + w^*_{max}) \le E(0, t + w^*(t))$$

and substituting this inequality into (67) we get

$$\dot{w}^*(t) \le \frac{\lambda}{w^*(t)E_{inf}} - 1. \tag{113}$$

Using Lemma 15 we get

$$\hat{t} := T^g + \frac{\lambda}{E_{inf}}\log\left(\frac{E_{inf}T^g}{\lambda}\right) \le t \le T \quad \Longrightarrow \quad w^*(t) \le 2\frac{\lambda}{E_{inf}}.$$

Define

$$z(t) := \frac{w^*(t)E(0, t + w^*(t))}{\lambda} - 1.$$

Using (67) we get

$$\dot{z}(t) = -\frac{1}{w^*(t)}z(t) + \frac{E'(0, t + w^*(t))}{E(0, t + w^*(t)}. \tag{114}$$

For $\hat{t} \le t \le T$ we have

$$-1 \le z(\hat{t}) \le 2\frac{E_{sup}}{E_{inf}}, \qquad \frac{1}{w^*(t)} \ge \frac{1}{2}\frac{E_{inf}}{\lambda}, \qquad \left|\frac{E'(0, t + w^*(t))}{E(0, t + w^*(t)}\right| \le \frac{D}{E_{inf}} \tag{115}$$

with the D of (65). Solving the linear ODE (114) and using the inequalities (115) we get

$$|z(t)| \le 2\frac{E_{sup}}{E_{inf}} \exp\left(-\frac{1}{2}\frac{E_{inf}}{\lambda}(t - \hat{t})\right) + \lambda\frac{2D}{E_{inf}^2}.$$

Thus for $t \ge \hat{t} + \frac{2}{E_{inf}}\lambda \log(\frac{1}{\lambda})$ we have $|z(t)| = \mathcal{O}(\lambda)$, which implies

$$w^*(t) - \frac{\lambda}{E(0, t + w^*(t))} = \mathcal{O}(\lambda^2).$$

If we combine this with

$$\left|\frac{\lambda}{E(0, t + w^*(t))} - \frac{\lambda}{E(0, t)}\right| \le \lambda^2 2\frac{D}{E_{inf}^3}$$

the claim of the Lemma follows. □

From this $\lambda m_1^\lambda(t) - E(0, t) = \varphi_\lambda(t) - \varphi_{crit}(t) = \mathcal{O}(\lambda)$ follows which proves (31). Now we prove (33) using Laplace transforms:

Lemma 17 *Let $U_\lambda(t, x)$ be the solution of (42) with a fixed initial condition $U(0, x)$ obtained from $\underline{\mathbf{v}}(0) \in \mathbf{V}^*$ and $\lambda(t) \equiv \lambda$. Then for any $t > T^g$ we have*

$$\lim_{\lambda\to 0} \frac{U'_\lambda(t, \frac{\lambda^2}{2E(0,t)}x)}{U'_\lambda(t, 0)} = \frac{1}{\sqrt{1 + x}}. \tag{116}$$

Proof Fix $\lambda > 0$ and denote the solution of (42) with $\lambda(t) \equiv \lambda$ by $U(t, x)$. For all $t \ge 0$ we have

$$X''(t, u) \ge \frac{1}{E_{sup}} \quad \Longrightarrow \quad X(t, u) \ge \frac{1}{2E_{sup}}u^2 \quad \Longrightarrow \quad |U(t, x)| = \mathcal{O}(\sqrt{x}).$$

We use the shorthand notation $E = E(0, t + w^*(t))$.

$$X'(t, u) = -w^*(t) + \frac{u}{E} + \mathcal{O}(u^2), \qquad X(t, u) = -uw^*(t) + \frac{u^2}{2E} + \mathcal{O}(u^3),$$

$$\begin{aligned} U(t, x) &= Ew^*(t) - \sqrt{(Ew^*(t))^2 + 2E(x - \mathcal{O}(U(t, x)^3))} \\ &= Ew^*(t) - \sqrt{(Ew^*(t))^2 + 2Ex} + \mathcal{O}(x), \end{aligned}$$

$$U'(t,x) = \frac{1}{X'(t,U(t,x))} = \frac{1}{-w^*(t) + \frac{U(t,x)}{E}} + \mathcal{O}(1)$$
$$= \frac{-1}{\sqrt{w^*(t)^2 + \frac{2}{E}x + \mathcal{O}(x)}} + \mathcal{O}(1) = \frac{-1}{\sqrt{w^*(t)^2 + \frac{2}{E}x}} + \mathcal{O}(1).$$

Because of Lemma 16 we have

$$\lim_{\lambda \to 0} \frac{\lambda^2}{E(0, t + w^*_\lambda(t))E(0,t)w^*_\lambda(t)^2} = 1$$

from which the claim of this lemma follows. □

The r.h.s. of (116) is the Laplace transform of the $\Gamma(\frac{1}{2},1)$ distribution and the r.h.s. of (33) is the distribution function of the $\Gamma(\frac{1}{2},1)$ distribution, so (33) follows from the continuity theorem of Laplace transforms.

Proof of Theorem 3 First observe that instead of proving uniform convergence of Φ_n to Φ_{crit} we only need to show convergence on $[0,T]$ for any T, because

$$T \geq T^g \implies m_0(T) = \int_{T+w^*(T)}^{\infty} E(0,w)dw \leq \int_T^\infty \frac{1}{w^2}dw = \frac{1}{T}$$

by (64), thus $0 \leq \Phi_n(t) - \Phi_{crit}(t) \leq \frac{1}{T}$ for $t \geq T$. If we prove that $w^*(t)$ is small for $t \geq \frac{1}{m_1(0)}$ then we are done by (70) and Lemma 5, since

$$\begin{aligned} &0 \leq \Phi_n(t) - \Phi_{crit}(t) = F(0, t + w_n^*(t)) - F(0,t) \leq w_n^*(t)E_{sup}, \quad t \geq T^g, \\ &\Phi_n(t) \leq \Phi_n(T^g) = F(0, T^g + w_n^*(T^g)) \leq w_n^*(T^g)E_{sup}, \quad t \leq T^g. \end{aligned} \tag{117}$$

We can give an upper bound on $w^*(t)$ for $t \geq T^g$ if we replace $\lambda(t)$ with λ_{sup} in (67): using (113) we get $w^*(t) = \mathcal{O}\left(\lambda \log(\frac{1}{\lambda})\right)$ if we substitute $t \geq T^g$ and $c = \frac{\lambda_{sup}}{E_{inf}}$ into (112), thus $\lim_{n\to\infty} w_n^*(t) = 0$ uniformly for $T^g \leq t \leq T$.

We obtain $\lim_{n\to\infty} v_k^n(t) = v_k(t)$ for $k = 1, 2, \ldots$ by the uniform convergence of $m_0^n(t)$ and $\lambda^n(t)$ to the critical $m_0(t) = m_0(0) - \Phi(t)$ and $\lambda(t) \equiv 0$ in (16). □

7 Proof of the Alternating Limit Theorem

We turn our attention to the proof of Theorem 4 and Theorem 9.

In this section we assume $m_0(0) = 1$ but the results generalize easily to the $m_0(0) \neq 1$ case, since if $\underline{\mathbf{v}}(t)$ is the solution of (19) with burning times $T_1^b, T_2^b, \ldots$ then $m_0(0)\underline{\mathbf{v}}(m_0(0)t)$ is also a solution of (19) with burning times $\frac{T_1^b}{m_0(0)}, \frac{T_2^b}{m_0(0)}, \ldots$.

Definition 13 If $\underline{\mathbf{v}}(t)$ is a solution of (19), let $w_+^*(t) := \frac{1}{m_1(t)}$.

If $w^*(t) \geq 0$ then $w_+^*(t) = w^*(t)$, but if $w^*(t) < 0$ then $w_+^*(t) = -X'(t, -\theta(t))$.

If t is a burning time then $w^*(t_+) := \lim_{\varepsilon\to 0} w^*(t+\varepsilon) = w_+^*(t)$.

Lemma 18 *We consider the solution of* (19) *on* $[0, T]$ *with an arbitrary sequence of burning times. If* $T^g \le t \le T$ *and* $w^*(t) < 0$ *then*

$$\theta(t) \ge \frac{m_1(0)}{m_2(0)} \frac{1}{T^2} \left| w^*(t) \right|, \tag{118}$$

$$w_+^*(t) \le 4\sqrt{\frac{m_2(0)}{m_1(0)}} \exp\left(\frac{m_2(0)}{m_1(0)} T + 1\right) \cdot \left| w^*(t) \right| =: C(T, \underline{\mathbf{v}}(0)) \left| w^*(t) \right|. \tag{119}$$

If $w^*(t) < 0$ *and if*

$$w_1 + \left| w^*(t) \right| \le t \le w_2 - \sqrt{\frac{E_{sup}(w_1, w_2)}{E_{inf}(w_1, w_2)}} \left| w^*(t) \right| \tag{120}$$

holds then

$$-\sqrt{\frac{E_{inf}(w_1, w_2)}{E_{sup}(w_1, w_2)}} w^*(t) \le w_+^*(t) \le -\sqrt{\frac{E_{sup}(w_1, w_2)}{E_{inf}(w_1, w_2)}} w^*(t), \tag{121}$$

$$-2E_{inf}(w_1, w_2) w^*(t) \le \theta(t) \le -2E_{sup}(w_1, w_2) w^*(t). \tag{122}$$

Proof By (49), $w_+^*(t) = -X'(t, -\theta(t))$, (109) and (64) we get

$$\theta(t) = F(t, w_+^*(t)) \ge \int_{w^*(t)}^{0} \frac{m_1(0)}{m_2(0)} \frac{1}{(t+y)^2} dy \ge \frac{m_1(0)}{m_2(0)} \frac{1}{T^2} \left| w^*(t) \right|.$$

Rearranging (50) and using (49) we get that $w = w_+^*(t)$ is the positive root of the function

$$f(w) := G(t, w) - F(t, w)w = G(t, 0) + \left(- \int_0^w y E(t, y) \right) dy = f(0) + (f(w) - f(0)).$$

We prove (119) by considering the cases $\frac{|w^*(t)|}{t} \le \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}}$ and $\frac{|w^*(t)|}{t} > \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}}$ separately.

If $\frac{|w^*(t)|}{t} \le \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}}$, then we prove that $w_+^*(t) \le 2\sqrt{\frac{m_2(0)}{m_1(0)}} |w^*(t)|$ by showing that $f(0) \le |f(w) - f(0)|$ with $w = 2\sqrt{\frac{m_2(0)}{m_1(0)}} |w^*(t)|$.

$$f(0) = \int_{w^*(t)}^{0} (-y) E(0, t+y) dy \le \int_0^{|w^*(t)|} \frac{y}{(t-y)^2} dy$$

by (109) and (64).

$$|f(w) - f(0)| \ge \int_0^w \frac{m_1(0)}{m_2(0)} \frac{y}{(t+y)^2} dy = \int_0^{|w^*(t)|} \frac{m_1(0)}{m_2(0)} \frac{y}{(t \frac{|w^*(t)|}{w} + y)^2} dy. \tag{123}$$

It is straightforward to check that

$$0 \le y \le \left| w^*(t) \right| \;\&\; \frac{|w^*(t)|}{t} \le \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}} \quad \Longrightarrow \quad \frac{y}{(t-y)^2} \le \frac{m_1(0)}{m_2(0)} \frac{y}{(t \frac{|w^*(t)|}{w} + y)^2}$$

which is sufficient for $f(0) \le |f(2\sqrt{\frac{m_2(0)}{m_1(0)}} |w^*(t)|) - f(0)|$ to hold.

If $\frac{|w^*(t)|}{t} > \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}}$, then

$$f(0) = G(t,0) = \int_{w^*(t)}^{0} F(t,y)dy \leq \left|w^*(t)\right| \leq T$$

since by (52) we have $F(t,y) \leq m_0(t) \leq m_0(0) = 1$. Calculating the middle integral in (123) we get that in order to have $f(0) \leq |f(w) - f(0)|$

$$\frac{m_1(0)}{m_2(0)}\left(\log\left(1 + \frac{w}{t}\right) - 1\right) \geq T$$

is sufficient. Rearranging this and using $\frac{|w^*(t)|}{t} > \frac{1}{4}\sqrt{\frac{m_1(0)}{m_2(0)}}$ we obtain (119).

The proof of the upper bound of (121) is similar: using (109) we get that $w_1 \leq t - |w^*(t)| \leq t + w \leq w_2$ implies

$$f(0) \leq \frac{1}{2}E_{sup}(w_1,w_2)w^*(t)^2, \qquad f(w) - f(0) \leq -\frac{1}{2}E_{inf}(w_1,w_2)w^2.$$

Using (120) the inequality $f(-\sqrt{\frac{E_{sup}(w_1,w_2)}{E_{inf}(w_1,w_2)}}w^*(t)) \leq 0$ follows. The lower bound of (121) is verified similarly.

If $u \in [-\theta(t), 0]$, then

$$X(t,u) \leq -w^*(t)u + \frac{1}{2}\frac{1}{E_{inf}(w_1,w_2)}u^2,$$

since $X''(t,u)$ with $u \in [-\theta(t), 0]$ is equal to $\frac{1}{E(0,t+y)}$ for some

$$y \in [w^*(t), w_+^*(t)] \subseteq \left[w^*(t), -\sqrt{\frac{E_{sup}(w_1,w_2)}{E_{inf}(w_1,w_2)}}w^*(t)\right],$$

thus $t + y \in [w_1, w_2]$ by (120). This implies the lower bound of (122), and the proof of the upper bound is similar. □

The proof of Theorem 4 is similar that of Theorem 3: if $\varepsilon = \sup_i\{T_{i+1}^b - T_i^b\}$ and $T_i^b < t \leq T_{i+1}^b$ then $w^*(t) = w_+^*(T_i^b) - (t - T_i^b) \geq -\varepsilon$ and by (119) we have $w_+^*(T_i^b) = \mathcal{O}(|w^*(T_i^b)|) = \mathcal{O}(\varepsilon)$ on $[0,T]$.

Lemma 19 *We consider the solution of* (19) *with initial critical core* $E(0,w)$. *If* $T_1^g < T_2^g$ *are two consecutive gelation times, then the unique burning time in between* T_1^g *and* T_2^g *is*

$$T^b(T_1^g, T_2^g) = \frac{\int_{T_1^g}^{T_2^g} yE(0,y)dy}{\int_{T_1^g}^{T_2^g} E(0,y)dy}. \tag{124}$$

Moreover

$$\theta(T^b(T_1^g, T_2^g)) = \int_{T_1^g}^{T_2^g} E(0,y)dy. \tag{125}$$

Proof T^b needs to satisfy $T_2^g - T^b = w_+^*(T^b)$, but by the proof of Lemma 18 $w_+^*(T^b)$ is the unique positive root of $G(T^b, 0) - \int_0^w yE(0, T^b + y)dy$. $G(T^b, 0) = -\int_{T_1^g - T^b}^0 yE(0, T^b + y)dy$ by (109), so $\int_{T_1^g - T^b}^{T_2^g - T^b} yE(0, T^b + y)dy = 0$ must hold, from which (124) easily follows.

By (49), $w_+^*(t) = -X'(t, -\theta(t))$ and (109) we get

$$\theta(T^b) = F(T^b, w_+^*(T^b)) = \int_{w^*(T^b)}^{w_+^*(T^b)} E(0, T^b + y)dy = \int_{T_1^g}^{T_2^g} E(0, y)dy, \qquad \square$$

Definition 14 If $\underline{\mathbf{v}}(t)$ is the solution of the random alternating equations (see Definition 6), denote by $T_1^b < T_2^b < \cdots$ the sequence of random burning times and by $T^g = T_1^g < T_2^g < \cdots$ the sequence of random gelation times. Indeed $T_1^g < T_1^b < T_2^g < T_2^b < \cdots$.

Let $\tau_i := T_{i+1}^g - T_i^g$ be the length of the i-th critical interval.

$$N(t) := \max\{i : T_i^g < t\}, \qquad \tau(t, i) := \tau_{N(t)+i},$$

$\tau(t, 0)$ is the length of the critical interval containing t.

Let $\theta(t, i) := \theta(T_{N(t)+i}^b)$, thus $\theta(t, 1)$ is the frozen mass of the first giant component born after t.

$$w_-^*(t, i) := T_{N(t)+i}^b - T_{N(t)+i}^g = -w^*(T_{N(t)+i}^b),$$
$$w_+^*(t, i) := T_{N(t)+i+1}^g - T_{N(t)+i}^b = w_+^*(T_{N(t)+i}^b).$$

Definition 15 A nonnegative random variable X has Rayleigh distribution with parameter σ, briefly $X \sim R(\sigma)$, if

$$\mathbf{P}(X > x) = \exp\left(-\frac{1}{2\sigma^2}x^2\right) =: R(\sigma, x),$$

$\mathbf{E}(X) = \sigma\sqrt{\frac{\pi}{2}}$. Y has a size-biased Rayleigh distribution with parameter σ, briefly $Y \sim R_{sb}(\sigma)$ if

$$\mathbf{P}(Y > y) = \frac{\mathbf{E}(X \cdot \mathbb{I}[X > y])}{\mathbf{E}(X)} = R_{sb}(\sigma, y).$$

The scaling identities

$$R(\sigma, x) = R(a\sigma, ax) \quad \text{and} \quad R_{sb}(\sigma, x) = R_{sb}(a\sigma, ax) \tag{126}$$

are valid for $a > 0$.

The r.h.s of (34) is $R_{sb}(\frac{1}{\sqrt{2}}, x)$.

The Rayleigh distribution emerges in our setting in the following way: if we consider the solution of the random alternating equations with burning times defined by a homogenous Poisson process with rate λ, forget about the error terms in (122) by assuming $w_1 = w_2$ then $\theta(t) = 2E \cdot (t - T_i^g)$ if $T_i^g < t \leq T_i^b$, so

$$\mathbf{P}\left(T_i^b - T_i^g > w\right) = \exp\left(-\lambda \int_0^w 2Esds\right) = R\left(\frac{1}{\sqrt{2E\lambda}}, w\right).$$

From $\theta(T_i^b) = 2E \cdot (T_i^b - T_i^g)$ and (126) we get $\theta(T_i^b) \sim R(\sqrt{\frac{2E}{\lambda}})$. Assuming $w_1 = w_2$ in (121) we get $w_+^*(T_i^b) = -w^*(T_i^b)$, thus $\tau_i \sim R(\sqrt{\frac{2}{E\lambda}})$.

Lemma 20 *If* $\underline{\mathbf{v}}(t)$ *is the solution of the random alternating equations with constant rate function* $\lambda(t) \equiv \lambda$ *then for every* $T^g \le t \le T$ *we have*

$$\mathbf{E}\left(\theta(T^b_{N(t)})\mathbb{I}[T^b_{N(t)} < T]\right) = \mathcal{O}(\lambda^{-\frac{1}{2}}), \tag{127}$$

$$\mathbf{E}\left(T^g_{N(t)+1} \wedge T - t\right) = \mathcal{O}(\lambda^{-\frac{1}{2}}) \tag{128}$$

as $\lambda \to \infty$ *where the constant in the* $\mathcal{O}$ *depends only on the initial data and* T.

Proof Let $\gamma(t) := t - T^g_{N(t)}$. Then

$$\begin{aligned}
&\lim_{dt\to 0} \frac{1}{dt}\mathbf{E}\left(\gamma(t+dt) - \gamma(t) \mid \mathcal{F}_t\right) \\
&= 1 - \gamma(t) \lim_{dt\to 0} \frac{1}{dt}\mathbf{P}\left(t \le T^g_{N(t)+1} \le t + dt \mid \mathcal{F}_t\right) \\
&= 1 - \gamma(t) \lim_{dt\to 0} \frac{1}{dt}\mathbf{P}\left(T^b(T^g_{N(t)}, t) \le T^b_{N(t)} \le T^b(T^g_{N(t)}, t+dt) \mid \gamma(t)\right) \\
&= 1 - \gamma(t)\theta(T^b(T^g_{N(t)}, t))\lambda \left.\frac{d}{ds}T^b(T^g_{N(t)}, s)\right|_{s=t} \\
&= 1 - \lambda E(0,t)\gamma(t)\left(t - T^b(t-\gamma(t), t)\right) \le 1 - \frac{1}{2}\lambda\frac{E_{inf}(T)^2}{E_{sup}}\gamma(t)^2
\end{aligned}$$

by Lemma 19. Taking the expectation of both sides of the above inequality and applying Jensen's inequality we get

$$\frac{d}{dt}\mathbf{E}\left(\gamma(t)\right) \le 1 - \frac{1}{2}\lambda\frac{E_{inf}(T)^2}{E_{sup}}\mathbf{E}\left(\gamma(t)\right)^2.$$

This differential inequality together with $\gamma(T^g) = 0$ implies

$$\mathbf{E}\left(\gamma(t)\right) \le \frac{1}{\sqrt{\lambda}}\frac{\sqrt{2E_{sup}}}{E_{inf}(T)} = \mathcal{O}(\lambda^{-\frac{1}{2}}), \quad T^g \le t \le T$$

by a "forbidden region"-type argument. Now we prove

$$\mathbf{E}\left(T^g_{N(t)+1} \wedge T - T^g_{N(t)}\right) = \mathcal{O}(\lambda^{-\frac{1}{2}}) \tag{129}$$

from which (128) trivially follows. We obtain (127) using (129) and $\theta(T^b_{N(t)}) \le 2E_{sup} \cdot (T^b_{N(t)} - T^g_{N(t)})$ by the upper bound of (122).

$$\begin{aligned}
T^g_{N(t)+1} \wedge T - T^g_{N(t)} &= \gamma(t) + \left(T^g_{N(t)+1} \wedge T - t\right)\mathbb{I}[t \ge T^b_{N(t)}] \\
&\quad + \left(T^g_{N(t)+1} \wedge T - T^b_{N(t)} \wedge T\right)\mathbb{I}[t < T^b_{N(t)}] \\
&\quad + \left(T^b_{N(t)} \wedge T - t\right)\mathbb{I}[t < T^b_{N(t)}],
\end{aligned}$$

$$\left(T^g_{N(t)+1}\wedge T - t\right)\mathbb{I}[t\geq T^b_{N(t)}]\leq w^*_+(t,0)\mathbb{I}[t\geq T^b_{N(t)}]$$
$$\leq C(T,\underline{\mathbf{v}}(0))w^*_-(t,0)\mathbb{I}[t\geq T^b_{N(t)}]\leq C(T,\underline{\mathbf{v}}(0))\gamma(t),$$

where $C(T,\underline{\mathbf{v}}(0))$ is defined in (119).

$$\left(T^g_{N(t)+1}\wedge T - T^b_{N(t)}\wedge T\right)\mathbb{I}[t< T^b_{N(t)}]\leq w^*_+(t,0)\mathbb{I}[t< T^b_{N(t)}\leq T]$$
$$\leq C(T,\underline{\mathbf{v}}(0))\gamma(t)$$
$$+C(T,\underline{\mathbf{v}}(0))\left(T^b_{N(t)}\wedge T - t\right)\mathbb{I}[t< T^b_{N(t)}].$$

By (22) and (118) we have

$$\begin{aligned}
&\mathbf{E}\left(\left(T^b_{N(t)}\wedge T - t\right)\mathbb{I}[t< T^b_{N(t)}]\right)\\
&\quad=\mathbf{E}\left(\left(T^b_{N(t)}\wedge T - t\right)\vee 0\right)\\
&\quad=\int_0^{T-t}\mathbf{P}\left(T^b_{N(t)}-t\geq x\right)dx\leq\int_0^{T-t}\exp\left(-\lambda\int_0^x\frac{m_1(0)}{m_2(0)}\frac{1}{T^2}y\,dy\right)=\mathcal{O}(\lambda^{-\frac12}),
\end{aligned}$$

$$\mathbf{E}\left(T^g_{N(t)+1}\wedge T - T^g_{N(t)}\right)=\mathcal{O}(\mathbf{E}\left(\gamma(t)\right))+\mathcal{O}\left(\mathbf{E}\left(\left(T^b_{N(t)}\wedge T-t\right)\vee 0\right)\right)=\mathcal{O}(\lambda^{-\frac12}).\qquad\square$$

Sketch proof of Theorem 9 Our aim is to make the following argument rigorous: Let

$$n(\lambda):=\left\lfloor\delta(\lambda)\sqrt{\frac{E(t,0)\lambda}{\pi}}\right\rfloor.$$

If $1\ll\lambda$ then $\theta(t,1),\theta(t,2),\dots,\theta(t,n(\lambda))$ are "almost" i.i.d. with distribution $\theta(t,i)\sim R(\sqrt{\frac{2E(t,0)}{\lambda}})$. $\tau(t,i)\approx\frac{\theta(t,i)}{E(t,0)}$, so

$$\sum_{i=1}^{n(\lambda)}\tau(t,i)\approx\delta(\lambda)$$

by the weak law of large numbers. Substituting $\hat{x}=2\sqrt{\frac{E(t,0)}{\lambda}}x$ into

$$\frac{\Phi\left(t+\delta(\lambda),\hat{x}\right)-\Phi\left(t,\hat{x}\right)}{\delta(\lambda)E(t,0)}\approx\frac{\sum_{i=1}^{n(\lambda)}\theta(t,i)\cdot\mathbb{I}[\theta(t,i)>\hat{x}]}{\sum_{i=1}^{n(\lambda)}\theta(t,i)}\approx\frac{\mathbf{E}\left(\theta(t,1)\mathbb{I}[\theta(t,1)>\hat{x}]\right)}{\mathbf{E}\left(\theta(t,1)\right)}$$

we get (34). $\square$

Proof of Theorem 9 We use the notations of Definitions 14 and 15.

$$E:=E(t,0)=E(0,t)=\varphi_{crit}(t).$$

We fix $x\geq 0$ and define

$$\hat{x}:=2\sqrt{\frac{E}{\lambda}}x,\qquad\theta(t,i,\hat{x}):=\theta(t,i)\mathbb{I}[\theta(t,i)>\hat{x}],\qquad n(\lambda,z):=\left\lfloor\delta(\lambda)\sqrt{\frac{E\lambda}{\pi}}(1+z)\right\rfloor.$$

By the assumption $\lambda^{-\frac12}\ll\delta(\lambda)$ we have $\lim_{\lambda\to\infty}n(\lambda,z)=+\infty$ for any $-1<z$.

Let $m(\lambda) := N(t+\delta(\lambda)) - N(t) - 1$.

$$\begin{aligned}\Phi\left(t+\delta(\lambda),\hat{x}\right)-\Phi\left(t,\hat{x}\right)=\theta(t,0,\hat{x})\mathbb{I}[T^b_{N(t)}>t]&\\ +\sum_{i=1}^{m(\lambda)}\theta(t,i,\hat{x})&\\ +\theta(t+\delta(\lambda),0,\hat{x})\mathbb{I}[T^b_{N(t+\delta(\lambda))}<t+\delta(\lambda)].&\end{aligned} \tag{130}$$

In order to prove (34) we only need to show that we have $\lim_{\lambda\to\infty}\mathbf{P}(B(\lambda,\varepsilon))=1$ for every $\varepsilon>0$ where

$$B(\lambda,\varepsilon):=\left\{R_{sb}(\frac{1}{\sqrt{2}},x)-\varepsilon<\frac{\sum_{i=1}^{m(\lambda)}\theta(t,i,\hat{x})}{E\delta(\lambda)}<R_{sb}(\frac{1}{\sqrt{2}},x)+\varepsilon\right\}$$

because the first and the last term on the r.h.s. of (130) divided by $E\delta(\lambda)$ converge to 0 in probability as $\lambda\to\infty$ by (127) and $\lambda^{-\frac{1}{2}}\ll\delta(\lambda)$.

$$E_{sup}(\lambda):=E_{sup}(t,t+2\delta(\lambda)),\qquad E_{inf}(\lambda):=E_{inf}(t,t+2\delta(\lambda)).$$

By (66) we have

$$\begin{gathered}E_{sup}(\lambda)\le E+2D\delta(\lambda)\quad\text{and}\quad E-2D\delta(\lambda)\le E_{inf}(\lambda),\\ C^u(\lambda):=1+\sqrt{\frac{E_{sup}(\lambda)}{E_{inf}(\lambda)}}\quad C^l(\lambda):=1+\sqrt{\frac{E_{inf}(\lambda)}{E_{sup}(\lambda)}},\end{gathered} \tag{131}$$

$\lim_{\lambda\to\infty}C^u(\lambda)=\lim_{\lambda\to\infty}C^l(\lambda)=2$, since $\delta(\lambda)\ll 1$.

We are going to couple the random variables $T^g_{N(t)+1}, w^*_-(t,1), w^*_-(t,2),\dots$ to

$$w^l_-(1), w^l_-(2),\dots\quad\text{and}\quad w^u_-(1), w^u_-(2),\dots,$$

where $w^l_-(i)\sim R(\frac{1}{\sqrt{2E_{sup}(\lambda)\lambda}})$ are i.i.d. and $w^u_-(i)\sim R(\frac{1}{\sqrt{2E_{inf}(\lambda)\lambda}})$ are i.i.d., moreover the auxiliary random variables are independent from $T^g_{N(t)+1}$. If we define the events

$$A^u(\lambda,z,z_2):=\left\{T^g_{N(t)+1}+C^u(\lambda)\cdot\sum_{j=1}^{n(\lambda,z)}w^u_-(j)\le t+\delta(\lambda)\cdot(1+z_2)\right\},$$

$$A^l(\lambda,z,z_2):=\left\{T^g_{N(t)+1}+C^l(\lambda)\cdot\sum_{j=1}^{n(\lambda,z)}w^l_-(j)\ge t+\delta(\lambda)\cdot(1+z_2)\right\}$$

then it is an easy consequence of (128), $\lambda^{-\frac{1}{2}}\ll\delta(\lambda)$, and the weak law of large numbers that

$$-1<z<z_2\implies\lim_{\lambda\to\infty}\mathbf{P}(A^u(\lambda,z,z_2))=1,$$

$$z>z_2>-1\implies\lim_{\lambda\to\infty}\mathbf{P}\left(A^l(\lambda,z,z_2)\right)=1.$$

Our coupling is going to satisfy

$$A^u(\lambda, z, 1) \subseteq \bigcap_{i=1}^{n(\lambda,z)} \{w_-^l(i) \le w_-^*(t,i) \le w_-^u(i)\} \tag{132}$$

for any z.

The joint construction of $w_-^l(j)$, $w_-^*(t,j)$ and $w_-^u(j)$ for $j = 1, 2, \dots$ is as follows: given $T^g_{N(t)+1}$ and $w_-^*(t,1), \dots, w_-^*(t, j-1)$ we can determine $T^g_{N(t)+j}$ by solving (19). For $s \ge 0$ Let

$$\mu(s) := \lambda\theta(T^g_{N(t)+j} + s), \qquad \mu_l(s) := \lambda 2E_{sup}(\lambda)s, \qquad \mu_u(s) := \lambda 2E_{inf}(\lambda)s.$$

Let $w_-^*(t,j)$, $w_-^l(j)$ and $w_-^u(j)$ be the horizontal coordinate of the leftmost point below the graphs of μ, μ_l and μ_u of the same standard uniform 2-dimensional Poisson process on the first quadrant of the plane. Thus $w_-^l(j) \sim R(\frac{1}{\sqrt{2E_{sup}(\lambda)\lambda}})$, $w_-^u(j) \sim R(\frac{1}{\sqrt{2E_{inf}(\lambda)\lambda}})$ are independent from everything that was constructed earlier and $\mathbf{P}(w_-^l(j) \le w_-^u(j)) = 1$. The joint distribution of $T^g_{N(t)+1}, w_-^*(t,1), \dots, w_-^*(t,j)$ agrees with that of the solution of the random alternating equation.

We are going to prove (132) by induction. Assume that $A^u(\lambda, z, 1)$ holds. If

$$\bigcap_{i=1}^{j-1}\{w_-^l(i) \le w_-^*(t,i) \le w_-^u(i)\} \cap \bigcap_{i=1}^{j-1}\{\tau(t,i) \le C^u(\lambda)\cdot w_-^u(i)\} \tag{133}$$

holds for some $j \le n(\lambda, z)$, then

$$T^g_{N(t)+j} = T^g_{N(t)+1} + \sum_{i=1}^{j-1}\tau(t,i) \le T^g_{N(t)+1} + C^u(\lambda)\sum_{i=1}^{j-1} w_-^u(i)$$

which implies $\mu_u(s) \le \mu(s) \le \mu_l(s)$ for $0 \le s \le w_-^u(j)$ by (122) and $A^u(\lambda, z, 1)$. From this $w_-^l(j) \le w_-^*(t,j) \le w_-^u(j)$ follows, and (121) can be applied to deduce

$$\tau(t,j) = w_-^*(t,j) + w_+^*(t,j) \le \left(1 + \sqrt{\frac{E_{sup}(\lambda)}{E_{inf}(\lambda)}}\right) w_-^*(t,j) \le C^u(\lambda) w_-^u(j).$$

Thus we can replace j with $j+1$ in (133). This completes the proof of (132). Let

$$\theta^u(t,i,\hat{x}) := 2E_{sup}(\lambda)w^u(i)\cdot \mathbb{I}[2E_{sup}(\lambda)w^u(i) > \hat{x}],$$

$$\theta^l(t,i,\hat{x}) := 2E_{inf}(\lambda)w^l(i)\cdot \mathbb{I}[2E_{inf}(\lambda)w^l(i) > \hat{x}].$$

(122) and (132) imply

$$A^u(\lambda, z, 1) \subseteq \bigcap_{i=1}^{n(\lambda,z)} \{\theta^l(t,i,\hat{x}) \le \theta(t,i,\hat{x}) \le \theta^u(t,i,\hat{x})\},$$

$$B^u(\lambda, z, \varepsilon) := \left\{ \frac{\sum_{i=1}^{n(\lambda,z)} \theta^u(t,i,\hat{x})}{E\delta(\lambda)} \le R_{sb}\left(\frac{1}{\sqrt{2}}, x\right) + \varepsilon \right\},$$

$$B^l(\lambda, z, \varepsilon) := \left\{ \frac{\sum_{i=1}^{n(\lambda,z)} \theta^l(t,i,\hat{x})}{E\delta(\lambda)} \ge R_{sb}\left(\frac{1}{\sqrt{2}}, x\right) - \varepsilon \right\}.$$

The law of large numbers, (126) and (131) imply that

$$z < \varepsilon \implies \lim_{\lambda\to\infty} \mathbf{P}\left(B^u(\lambda, z, \varepsilon)\right) = 1 \quad \text{and} \quad -\varepsilon < z \implies \lim_{\lambda\to\infty} \mathbf{P}\left(B^l(\lambda, z, \varepsilon)\right) = 1.$$

We can use (132) and (121) to show

$$A^u(\lambda, z, 1) \subseteq \bigcap_{i=1}^{n(\lambda,z)} \{C^l(\lambda) w_-^l(i) \le \tau(t,i) \le C^u(\lambda) w_-^u(i)\}.$$

Since

$$m(\lambda) = \max\left\{ j : T^g_{N(t)+1} + \sum_{i=1}^{j} \tau(t,i) < t + \delta(\lambda) \right\}$$

and $A^u(\lambda, z, 0) \subseteq A^u(\lambda, z, 1)$ by definition,

$$A^u(\lambda, z, 0) \subseteq \{m(\lambda) \ge n(\lambda, z)\}, \qquad A^l(\lambda, z, 0) \cap A^u(\lambda, z, 1) \subseteq \{m(\lambda) \le n(\lambda, z)\},$$

$$A^l\left(\lambda, \frac{\varepsilon}{2}, 0\right) \cap A^u\left(\lambda, \frac{\varepsilon}{2}, 1\right) \cap B^u\left(\lambda, \frac{\varepsilon}{2}, \varepsilon\right) \cap B^l\left(\lambda, -\frac{\varepsilon}{2}, \varepsilon\right) \cap A^u\left(\lambda, -\frac{\varepsilon}{2}, 0\right) \subseteq B(\lambda, \varepsilon).$$

This completes the proof of $\lim_{\lambda\to\infty} \mathbf{P}\left(B(\lambda, \varepsilon)\right) = 1$. $\square$

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